Published by Institute of Physics Publishing for SISSA/ISAS
Received: *September 13, 2004*
Accepted: *November 9, 2004*

Chiral rings, anomalies and electric-magnetic duality

Luca Mazzucato

International School for Advanced Studies (SISSA/ISAS)
Via Beirut 2 - 4, 34014 Trieste, Italy, and
INFN, sez. di Trieste, Italy
E-mail: mazzu@sissa.it

Abstract: We study electric-magnetic duality in the chiral ring of a supersymmetric $\mathrm{U}(N_c)$ gauge theory with adjoint and fundamental matter, in presence of a general confining phase superpotential for the adjoint and the mesons. We find the magnetic solution corresponding to both the pseudoconfining and Higgs electric vacua. By means of the Dijkgraaf-Vafa method, we match the effective glueball superpotentials and show that in some cases duality works exactly offshell. We give also a picture of the analytic structure of the resolvents in the magnetic theory, as we smoothly interpolate between different Higgs vacua on the electric side.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality, Supersymmetric Effective Theories, Matrix Models.

 http://jhep.sissa.it/archive/papers/jhep112004020/jhep112004020.pdf

JHEP11(2004)020

Contents

1. Introduction and summary **2**
- 1.1 Duality in SQCD with adjoint matter 3
- 1.2 Generic deformation of KSS and duality 4
- 1.3 Offshell vs. onshell duality 6
- 1.4 Outline of the paper 6

2. The electric pseudoconfining phase **7**
- 2.1 The classical vacua 7
- 2.2 The electric stability bound and magnetic instantons 12
- 2.3 The chiral ring 14
- 2.4 The effective actions 18
- 2.5 The offshell duality map 21
- 2.6 The generic pseudoconfining case 22

3. The electric Higgs phase **27**
- 3.1 The classical vacua 27
- 3.2 The chiral ring 30

4. The cubic superpotential **34**
- 4.1 The pseudoconfining case 34
- 4.2 The Higgs case 35

5. Discussion **37**

A. The DV method **38**

B. Some properties of the effective glueball superpotential **40**
- B.1 Properties of the h_i 40
- B.2 Evaluation of $\mathcal{W}_{\rm eff}$ 41

C. Several Higgs solution **42**
- C.1 Two-Higgs case 42
- C.2 Several Higgs 43

JHEP11(2004)020

1. Introduction and summary

Supersymmetric $\mathcal{N} = 1$ gauge theories with matter exhibit a generalization of electric-magnetic duality to the case of non-abelian gauge groups, as found by Seiberg in the nineties [1]. This kind of duality is not exact at all scales, but it holds at large distances only and it proves to be a powerful tool in understanding the infrared dynamics of gauge theories. Consider an asymptotically free supersymmetric gauge theory, that we will call *electric*, whose renormalization group flow has a fixed point at a long distance scale, where the physics is described by a superconformal field theory. The *magnetic* dual is another theory which flows to the same fixed point. In other words, the physics at the infrared point is described equivalently by both theories. The two sets of degrees of freedom of the dual pair might be in general very different at the level of the microscopic lagrangian, as duality holds only for the two low energy effective field theories.

There are basically two different methods to study the low energy superpotentials of $\mathcal{N} = 1$ supersymmetric gauge theories. We can use of the tools of holomorphy and symmetry to contrain the superpotential part of the low energy effective action, in such a way to uniquely fix it (for a review see [2]). Alternatively, in theories with confinement and a mass gap, another way to compute the effective glueball superpotential has been proposed by Dijkgraaf and Vafa (DV) [3] and further developed by Cachazo et al. [4], based on the solution of some generalized Konishi anomaly equations.

Let us see how these methods work in the prototypical example of electric-magnetic duality, namely Seiberg duality in SQCD with $\mathrm{U}(N_c)$ gauge group and N_f flavors of quarks Q^f and antiquarks $\widetilde{Q}_{\tilde{f}}$ [1]. If we want to apply DV, the theory has to be massive and we need a tree level mass term

$$W_{\rm tree} = m\widetilde{Q}_f Q^f \,. \tag{1.1}$$

Even if classically the mesons vanish, at the quantum level their expectation value is set by the Konishi anomaly to $\langle \widetilde{Q}_f Q^f \rangle = N_f S/m$, where S is the glueball superfield [5]. The effective glueball superpotential is recovered by integrating this exact expectation value with respect to the corresponding coupling. We have to add also possible coupling independent terms, that in this case are the Veneziano-Yankielowicz superpotential $N_c S(1-\log S)$ and the one-loop exact renormalization of the gauge field $(3N_c - N_f) S \log \Lambda$, obtaining the glueball effective superpotential

$$W_{\rm eff} = S\left(\log \frac{m^{N_f}\Lambda^{3N_c-N_f}}{S^{N_c}} + N_c\right). \tag{1.2}$$

The magnetic dual of this theory is a supersymmetric gauge theory with $\mathrm{U}(\bar{N}_c)$ gauge group, N_f flavors of magnetic quarks q_f and antiquarks $\widetilde{q}^{\tilde{f}}$ and N_f^2 gauge singlets $P^{f}_{\tilde{f}}$, that represent the electric mesons. The classical magnetic superpotential corresponding to (1.1) is $\overline{W}_{\rm tree} = \frac{1}{\mu}P\widetilde{q}q + \overline{m}\,\mathrm{tr}\,P$. In the magnetic theory we have two basic equations to solve. The first is the singlet equation of motion, which completely fix the magnetic mesons to $\langle \widetilde{q}q \rangle = -\mu\overline{m}$. Since the singlets are not coupled to the gauge fields, their equations of motion are exact in the chiral ring of the quantum theory. Then we have the Konishi

JHEP11(2004)020

anomaly, that sets $\langle P\rangle = -\bar{S}/\overline{m}$. The effective glueball superpotential is then computed as in the electric case and we get

$$\overline{W}_{\text{eff}} = \bar{S}\left(\log\frac{\bar{S}^{N_f-\bar{N}_c}\widetilde{\Lambda}^{3\bar{N}_c-N_f}}{(-\overline{m}\mu)^{N_f}} + (\bar{N}_c - N_f)\right). \tag{1.3}$$

To find the duality map, we first match the electric mesons with the magnetic singlets, since they are directly related by a Legendre transform, and we see that $S = -\bar{S}$ and $m = \overline{m}$. Then we match the effective glueball superpotential and find the relation between the gauge groups $\bar{N}_c = N_f - N_c$ as well as the scale matching relation $\Lambda^{3N_c-N_f}\widetilde{\Lambda}^{3\bar{N}_c-N_f} = (-)^{N_f-N_c}\mu^{N_f}$.

However, the DV method is not really necessary in this case.[1] We can easily obtain the onshell expectation values of chiral operators by studying the nonperturbative low energy superpotentials of electric and magnetic theories, without ever introducing the glueball superfield. On the other hand, we can also integrate in the glueball superfield to obtain directly the glueball effective superpotential. On the electric side, the low energy theory is just pure $\mathrm{U}(N_c)$ SYM, whose nonperturbative superpotential is $W_{\text{low}} = N_c(\Lambda_{\text{low}}^{3N_c})^{\frac{1}{N_c}}$, which is responsible for gaugino condensation. One first matches the low energy scale $\Lambda_{\text{low}}^{3N_c} = m^{N_f}\Lambda^{3N_c-N_f}$ and then just integrate in the glueball to obtain directly (1.2). On the other side, in the magnetic theory the singlet equations of motion force all the flavors to be higgsed, thus the low energy theory is pure SYM with gauge group $\mathrm{U}(\bar{N}_c - N_f)$, whose low energy superpotential is the same as the electric one but with the appropriate magnetic quantities instead. By matching the magnetic scales and again integrating in the glueball we obtain (1.3).

1.1 Duality in SQCD with adjoint matter

In this paper we will study a more complicated version of Seiberg duality, first analyzed by Kutasov, Schwimmer and Seiberg (KSS), namely a supersymmetric gauge theory with gauge group $\mathrm{U}(N_c)$ and N_f flavors of quarks Q^f and antiquarks $\widetilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group [7, 8, 9]. The magnetic dual of the theory without a superpotential is not known. But we can study deformations by relevant superpotential couplings, for which we know the dual theory. A way to simplify the dynamics, which was studied by KSS, is to add a generic polynomial superpotential for the adjoint[2]

$$\begin{aligned} W_{el} &= \operatorname{Tr} V(X)\,, \\ V(z) &= \sum_{k=1}^{n}\frac{t_k}{k+1}x^{k+1}\,, \end{aligned} \tag{1.4}$$

that drives the theory to a confining phase in the infrared, leaving at low energy no dynamics but rather just a discrete set of vacua.

[1]The matrix model approach to Seiberg duality has been first used in [6].

[2]These operators are usually referred to as dangerously irrelevant, meaning that they are irrelevant at the UV fixed point when $n \geq 3$, but they become relevant as we flow to the infrared.

The magnetic dual of the theory (1.4) is a supersymmetric gauge theory with gauge group $U(\bar{N}_c)$, where $\bar{N}_c = nN_f - N_c$, and N_f flavors of dual quarks $q_{\tilde{f}}$ and antiquarks $\tilde{q}^f$, an adjoint chiral superfield Y and N_f^2 gauge singlets $(P_j)^f_{\tilde{f}}$, $j = 1, \ldots, n$, that represent the electric mesons

$$P_j = \widetilde{Q} X^{j-1} Q \,.$$

The magnetic theory is defined by the tree level superpotential

$$W_{mag} = -\operatorname{Tr} V(Y) + \tilde{q}\widetilde{m}(P,Y)q \,, \tag{1.5}$$

where $\widetilde{m}(P,z)$ is a certain degree $n-1$ polynomial, whose coefficients depend on the gauge singlets P_j. This magnetic polynomial will be the crucial quantity to evaluate in the quantum theory. Even if classically the chiral rings and the vacua of the two theories are very different, quantum mechanically they coincide. In particular, KSS proposed that the dynamically generated scales Λ of the electric theory and $\widetilde{\Lambda}$ of the magnetic theory obey the matching relation

$$\Lambda^{2N_c - N_f} \widetilde{\Lambda}^{2\bar{N}_c - N_f} = \mu^{2N_f} t_n^{-2N_f} \,, \tag{1.6}$$

which is very similar to the corresponding scale matching of SQCD we discussed above, and they checked the matching against various flows.

1.2 Generic deformation of KSS and duality

The purpose of this paper is to generalize the analysis of KSS by considering the most generic electric superpotential, obtained by adding to (1.4) a meson deformation

$$\operatorname{Tr} V(X) + \widetilde{Q}_{\tilde{f}} m(X)^{\tilde{f}}_f Q^f \,, \tag{1.7}$$

where in the classical chiral ring the degree of the meson polynomial $m(z)$ is at most $n-1$. At the classical level, this electric theory presents two different kinds of vacua. In the first vacuum, that we denote as *pseudoconfining*, the fundamentals vanish and the adjoint acquires a vacuum expectation that drives the theory to a product of low energy $U(N_i)$ SQCD blocks such that $\sum_{i=1}^n N_i = N_c$. The other vacuum is called the *Higgs* vacuum and is characterized by a nonvanishing classical vev for the fundamentals.[3] In this case the rank of the gauge group decreases. If we higgs L colors, then the low energy theory is still a product of $U(N_i)$ SQCD blocks, but now $\sum_{i=1}^n N_i = N_c - L$.

Our first analysis of the duality will focus on the map between the electric and magnetic classical vacua in both the pseudoconfining and the Higgs phase. The magnetic dual of the theory (1.7) contains, in addition to the superpotential (1.5), a source term for the gauge singlets $\sum_{k=1}^{\deg m+1} m_k P_k$, where m_k are the coefficients of $m(z)$. The magnetic vacua will depend then on the details of the electric meson polynomial: each flavor appearing in $m(z)^{\tilde{f}}_f$ turns on a higgsed block in the magnetic adjoint $\langle Y \rangle$. In particular, we will study

[3] In presence of matter in the fundamental representation of the gauge group there is no phase transition between Higgs and confining regimes and in the quantum theory one can continuously interpolate between them.

the magnetic vacuum corresponding to the electric Higgs phase, characterized by a nonzero classical vev for the magnetic singlets P_j. In our classical solution, as we increase the higgsed directions in the electric theory, thus driving it to weaker coupling, the higgsed block in the magnetic theory decreases its rank, driving the dual theory to stronger coupling.

We will consider then the map between the chiral rings of the two quantum theories. Due to the presence of a large number of couplings in the tree level action (1.7), the study of the effective superpotentials by the conventional methods of holomorphy and symmetries is more involved in this case. Therefore, we found more convenient to analyze the quantum theory with the DV method, along the lines discussed above for SQCD. In particular, we will concentrate on the operators that generate the chiral ring

$$M(z) = \left\langle \widetilde{Q} \frac{1}{z - X} Q \right\rangle , \qquad T(z) = \left\langle \mathrm{Tr}\, \frac{1}{z - X} \right\rangle .$$

A generalized version of the Konishi anomaly allows us to solve explicitly for these operators as functions of the glueball superfield S and the couplings [10, 11] and we can integrate them to obtain the glueball effective superpotentials. By matching first the electric mesons with the magnetic singlets and then the two effective superpotentials, we will derive the relation between the two gauge groups $\bar{N}_c = nN_f - N_c$ as well as the scale matching (1.6) and the map between the electric and magnetic chiral ring operators. The low energy electric and magnetic theories will be both described by the same hyperelliptic Riemann surface $y^2 = V'(z)^2 + f(z)$, a double-sheeted cover of the plane, where the quantum deformation $f(z)$ is a degree $n - 1$ polynomial. The pseudoconfining and Higgs duality map will turn out to be rather different, though. In particular, in the electric pseudoconfining phase the magnetic anomaly equations are solved by the simple condition

$$m(a_i)\widetilde{m}(a_i) = f(a_i) \,, \tag{1.8}$$

for $i = 1, \ldots, n$, where a_i are the roots of $V'(z)$. This condition will ensure also the match of the electric and magnetic chiral rings and will reproduce the Konishi anomaly in each low energy SQCD block.

The DV method allows us to study also the rich analytic structure of the low energy effective theory. Even if the electric and magnetic theory have the same curve, the meromorphic functions $M(z)$ and $T(z)$ living on the curve have very different analytic structures on the two sides. We will picture their analytic behavior as follows. According to [11], an Higgs eigenvalue in the electric theory is seen as a pole of $M(z)$ on the first (semiclassical) sheet of the curve. As we will see, in the magnetic theory the corresponding $\widetilde{M}(z)$ will have $n - 1$ poles on the first sheet. We can higgs twice the electric theory by bringing a second pole of $M(z)$ from the second (invisible) sheet into the first one. The magnetic theory behaves in two different ways depending on whether we higgs different electric flavors or several times the same flavor.[4] We will see that, in this latter case, the second electric higgsing corresponds in the magnetic theory to moving one of the $n - 1$ poles away from the first into the second sheet.

[4]At most we can higgs $n - 1$ color directions on the same flavor, corresponding to the degree of the meson polynomial $m(z)$.

1.3 Offshell vs. onshell duality

Seiberg duality describes of an electric and a magnetic theory that, inside the conformal window, flow to the same fixed point in the infrared, where both theories are in a non-abelian Coulomb phase. The dual effective actions for the massless fields at the fixed point are equivalent. However, a necessary requirement for the DV method to apply is that all the fields have to be massive: both theories flow to just a discrete set of vacua in the far infrared. The effective glueball superpotentials we are computing with the DV method are offshell actions valid above the energy scale set by the mass of the glueballs. If the theory has a mass gap, a natural expectation would be that duality works onshell only, after integrating out all massive fields, unless we could make sense of an S-matrix for the glueballs. In other words, we would not really expect a change of variable in the matrix integral, that computes the electric glueball superpotential, to give us back the magnetic glueball superpotential. However, at the level of SQCD we have seen that Seiberg duality works exactly offshell, as will occur in the first case we will analyze, when the meson polynomial is just a mass term for all the flavors. When the meson polynomial is z-dependent, instead, offshell duality will hold only for the first term in the semiclassical expansion of the theory.

1.4 Outline of the paper

Our main concern will be to compare electric and magnetic results at every stage of the computation. For this reason, we will tackle separately the two electric pseudoconfining and Higgs vacua and in each of them we will match first the classical and then the quantum theories.

In section 2 we will consider the electric pseudoconfining vacuum in presence of the generic deformation (1.7). We will find the corresponding classical magnetic solution and see that this is only valid for a small number of massive electric flavors, due to the presence of instanton effects in the broken magnetic gauge group. We will then study the quantum chiral rings of the dual pair by the DV method. First we will consider the case in which the electric meson superpotential is just a mass term, and we will show that in this case duality works exactly offshell. For a generic meson polynomial, instead, the solution (1.8) that we found is not exact offshell, but still it reproduces the usual Konishi anomaly in the low energy SQCD blocks and we believe it to hold onshell.

We will consider then the electric Higgs phase in section 3 and follow the same steps of the previous section, first the classical and then the quantum analysis, gaining in this way a complete picture of duality in the different vacua. Even if the solution of the quantum theory in this case will be implicit, we will be able to sketch the analytic behavior of the magnetic resolvents when moving the poles between the two sheets in the electric theory.

In section 4 we will consider the case of cubic tree level superpotential to see how duality works in a specific example and finally, in section 5, we will speculate about some questions raised by our analysis.

In the appendices we postponed some details of our computation of the effective superpotential of section 2, which is different from the one in [11]. In the last appendix we show a classical magnetic solution that generalizes the ones in section 3 for the Higgs phase.

JHEP11(2004)020

2. The electric pseudoconfining phase

In this section we will see how electric-magnetic duality works in the electric pseudoconfining phase. Our notations will be as follows. In the classical analysis of sections 2.1 and 2.2, we will always use the electric couplings to describe the magnetic theory, assuming that we know the duality map. In the analysis of the quantum theory, from section 2.3 on, we will overline the magnetic couplings to avoid possible confusion and then derive the duality map.

2.1 The classical vacua

The electric theory. Let us set the stage for our calculations. We will consider an $\mathcal{N}=1$ supersymmetric gauge theory with $\mathrm{U}(N_c)$ gauge group, that we will call *electric*. The matter content consists of N_f flavors of quarks Q^f and antiquarks $\widetilde{Q}_{\tilde{f}}$ and a chiral superfield X in the adjoint representation of the gauge group. We will at first let the theory flow to its infrared superconformal fixed point. Then we will turn on the generic tree level superpotential

$$\begin{aligned} W_{el} &= \mathrm{Tr}\, V(X) + \widetilde{Q}_{\tilde{f}} m(X)^{\tilde{f}}_{f} Q^f \,, \\ V'(z) &= \sum_{i=1}^{n} t_i z^i \,, \\ m(z)^{\tilde{f}}_{f} &= \sum_{k=1}^{l+1} (m_k)^{\tilde{f}}_{f} z^{k-1} \,, \end{aligned} \tag{2.1}$$

which is irrelevant in the UV but becomes relevant in the infrared. It will be useful to parameterize the adjoint polynomial as $V'(z) = t_n \prod_{i=1}^{n}(z-a_i)$ in terms of its roots. We denote the roots of the meson polynomial $m(z)$ as x_k, for $k=1,\dots,l$. The degree of $m(z)$ is $l \le n-1$, since higher mesons are trivial in the classical chiral ring, that contains the following operators

$$\mathrm{Tr}\, X^j \,, \qquad \widetilde{Q} X^{j-1} Q \,, \tag{2.2}$$

for $j=1,\dots,n$, as well as operators of the kind $\mathrm{Tr}\, W_\alpha X^j$ and $\mathrm{Tr}\, W_\alpha W^\alpha X^j$. However, $W_\alpha Q^f$ and $\widetilde{Q}_{\tilde{f}} W_\alpha$ are not in the chiral ring. Also, since the gauge group is $\mathrm{U}(N)$ rather than $\mathrm{SU}(N)$ we do not include "baryonic operators". Our main attention will be focused on the following chiral operators, that generate the chiral ring

$$\begin{aligned} R(z) &= -\frac{1}{32\pi^2}\, \mathrm{Tr}\, \frac{W_\alpha W^\alpha}{z-X} \,, \\ M^{f}_{\tilde{f}}(z) &= \widetilde{Q}_{\tilde{f}} \frac{1}{z-X} Q^f \,, \\ T(z) &= \mathrm{Tr}\, \frac{1}{z-X} \,, \\ w_\alpha(z) &= \frac{1}{4\pi}\, \mathrm{Tr}\, \frac{W_\alpha}{z-X} \,. \end{aligned} \tag{2.3}$$

We will set to zero in the following $w_\alpha(z)$ since its duality properties are automatic and does not constrain the other results.

JHEP11(2004)020

This theory exhibits two kinds of classically distinct vacua, that we will call pseudoconfining and Higgs vacua. In this section we will be concerned only with the former and leave the analysis of the Higgs vacuum to the section 3. The pseudoconfining vacua are characterized by vanishing expectation values for the fundamentals

$$X = \begin{pmatrix} a_1 & & \\ & \cdot & \\ & & a_n \end{pmatrix}$$
$$\widetilde{Q}_{\widetilde{f}} = 0\,, \qquad Q^f = 0\,, \tag{2.4}$$

where each a_i has multiplicity N_i such that $\sum_i N_i = N_c$. The reason why these are called "pseudoconfining" rather than "confining" vacua is that, due to the presence of fields in the fundamental representation of the gauge group, there is no phase transition between these vacua and the Higgs ones and in the quantum theory they are continuously connected. At low energy the theory consists of a set of decoupled $\mathrm{U}(N_i)$ SQCD with N_f flavors, while the adjoint has been integrated out.[5] The rank of the gauge group does not decrease along this flow.

The magnetic theory. The magnetic theory corresponding to (2.1) is again an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group $\mathrm{U}(\bar{N}_c)$ and N_f flavors of dual quarks q_f and antiquarks $\widetilde{q}^{\widetilde{f}}$.[6] We also add a chiral superfield Y in the adjoint and N_f^2 gauge singlets $(P_j)^f_{\widetilde{f}}$, for $j = 1, \ldots, n$. We first let this theory flow to its interacting superconformal fixed point, then we add the following superpotential

$$\begin{aligned} W_{\mathrm{mag}} &= -\operatorname{Tr} V(Y) + \widetilde{q}\widetilde{m}(P,Y)\, q + \sum_{j=1}^{l+1} m_j P_j\,, \\ \widetilde{m}(z) &= \frac{1}{\mu^2}\sum_{k=1}^{n} t_k \sum_{j=1}^{k} P_j z^{k-j}\,, \end{aligned} \tag{2.5}$$

where we suppressed the flavor indices and $V(z)$ and the m_k's are the electric ones in (2.1).

We introduced the degree $n-1$ polynomial $\widetilde{m}(P,z)$, which can be conveniently cast in the form[7]

$$\widetilde{m}(z) = \frac{1}{\mu^2}\oint_A d\zeta \frac{V'(\zeta) - V'(z)}{\zeta - z} P(\zeta)\,, \tag{2.6}$$

where A is a contour that sorrounds all the roots of $V'(z)$. We introduced also a meromorphic function that collects for the gauge singlets

$$P(z) = P_1 z^{-1} + \cdots + P_n z^{-n}\,, \tag{2.7}$$

[5] If we allow for double roots in $V'(z)$ we end up with adjoint SQCD with a cubic tree level superpotential for the low energy adjoint superfield. For simplicity we will consider superpotential with only single roots, though.

[6] Note that $\widetilde{q}^{\widetilde{f}}$ is in the fundamental representation of the flavor symmetry group, while q_f is in the antifundamental.

[7] We will always understand a factor $\frac{1}{2\pi i}$ in the measure of the contour integrals.

and note that the last term in the superpotential (2.5) can be rewritten as

$$\oint_A m(z)P(z)\,. \tag{2.8}$$

Moreover, by inverting (2.6) we find that the general expression for the singlets is fixed by $\widetilde{m}(z)$ to

$$P(z) = \mu^2 \left[\frac{\widetilde{m}(z)}{V'(z)}\right]_{-n}, \tag{2.9}$$

meaning that we take the Laurent expansion up to $\mathcal{O}(z^{-n})$. The equations of motion for the singlets are

$$\begin{aligned} \sum_{i=j}^{n} t_i \widetilde{q} Y^{i-j} q &= -\mu^2 m_j\,, \qquad j = 1,\ldots,l\,, \\ \sum_{i=j}^{n} t_i \widetilde{q} Y^{i-j} q &= 0\,, \qquad j = l+1,\ldots,n\,. \end{aligned} \tag{2.10}$$

Therefore, the classical chiral ring of this theory does not contain the mesons $\widetilde{q}Y^{j-1}q$, which are in fact replaced by the n singlets P_j. We already see here that the analysis of the chiral ring in this theory will be slightly different than the usual electric one. We will be still interested in the following chiral operators

$$\begin{aligned} \widetilde{R}(z) &= -\frac{1}{32\pi^2}\,\mathrm{Tr}\,\frac{W_\alpha W^\alpha}{z-Y}\,, \\ \widetilde{M}_f^{\widetilde{f}}(z) &= \widetilde{q}^{\widetilde{f}} \frac{1}{z-Y} q_f\,, \\ \widetilde{T}(z) &= \mathrm{Tr}\,\frac{1}{z-Y}\,. \end{aligned} \tag{2.11}$$

We already set to zero the magnetic w_α generator analogous to the one in (2.3).

Now we want to look at the magnetic vacuum corresponding to the pseudoconfining electric one in (2.4). This phase is characterized by a vanishing classical expectation value for the gauge singlets P_j, since they represent to the electric mesons. We have to satisfy the singlet equations of motion (2.10), as well as the adjoint ones $V'(Y) = 0$. Consider at first the simple case in which only the last flavor appears in the electric meson superpotential (2.1), i.e. $m(z)_f^{\widetilde{f}} = m(z)_{N_f}^{N_f}$. Correspondingly, the right hand side of the singlet equations of motion (2.10) has nonvanishing entries only along these flavor directions. Let us denote

$$b_1 = \left(-\frac{m_1\mu^2}{t_n}\right)^{\frac{1}{n+1}}, \tag{2.12}$$

which has the dimension of a mass, and introduce the following bra-ket notation

$$|i\rangle \leftrightarrow i^\alpha = \delta_i^\alpha\,,$$

where a ket corresponds to a field in the fundamental representation of the gauge group and a bra to a field in the antifundamental. We introduce also the shift operator acting on the first n entries

$$R_n|i\rangle = \begin{cases} |i-1\rangle & i=2,\dots,n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

In this notations, the classical expectation value for the adjoint can be represented in block diagonal form

$$Y = \operatorname{diag}(Y_{N_f}, Y_{a_1}, \dots, Y_{a_n}), \tag{2.14}$$

where

$$\begin{aligned} Y_{N_f} &= |1\rangle\langle v_{N_f}| + b_1 R_n, \\ |v_{N_f}\rangle &= -\sum_{k=1}^{n-1} \frac{t_{n-k}}{t_n} b_1^{1-k}|k\rangle. \end{aligned} \tag{2.15}$$

The first $n \times n$ block (2.15) reads

$$Y_{N_f} = \begin{pmatrix} -\frac{t_{n-1}}{t_n} & b_1 & 0 & & & \\ -\frac{t_{n-2}}{t_n}\frac{1}{b_1} & 0 & b_1 & . & & \\ . & & 0 & . & . & \\ . & & & . & b_1 & 0 \\ -\frac{t_1}{t_n}\frac{1}{b_1^{n-2}} & 0 & . & . & 0 & b_1 \\ 0 & & & & . & 0 \end{pmatrix}. \tag{2.16}$$

The blocks $Y_{a_1}, \dots, Y_{a_n}$ in (2.14) correspond to the electric pseudoconfining eigenvalues and are given by

$$Y_{a_i} = \operatorname{diag}(a_i, \dots, a_i). \tag{2.17}$$

Each Y_{a_i} block has rank $\bar{N}_i = N_f - N_i - 1$. Note that this is different from the usual relation $\overline{N}_i = N_f - N_c$ that we have in each low energy Seiberg block when the meson polynomial $m(z)$ is switched off. We can check that with this solution the correct magnetic rank is reproduced. Since $\sum_i N_i = N_c$, we have

$$\sum_{i=1}^{n}(N_f - N_i - 1) + n = nN_f - N_c = \overline{N}_c. \tag{2.18}$$

The magnetic quarks are all vanishing except the last flavor

$$\begin{aligned} |\widetilde{q}^{N_f}\rangle &= b_1|1\rangle, \\ |q_{N_f}\rangle &= \sum_{i=1}^{l+1} b_1^i \frac{m_i}{m_1}|n+1-i\rangle, \end{aligned} \tag{2.19}$$

whose vevs are along the first n color directions, in order to sandwich the first block Y_{N_f} in the adjoint and satisfy the singlet equations of motion. Note that, in the simplest case in which $V'(z) = t_n z^n$ and $m(z)^f_{\widetilde{f}} = m^{N_f}_{N_f}$, this solution reduces to the usual KSS solution [8, 9]. We can also write down the classical expressions for the generators (2.11) for this particular solution. The resolvent $\widetilde{R}(z)$ vanishes while

$$
\begin{aligned}
\widetilde{M}_{\rm cl}(z)_{N_f}^{N_f} &= -\mu^2 \frac{m(z)}{V'(z)}\,, \\
\widetilde{T}_{\rm cl}(z) &= \frac{d}{dz}\ln\frac{V'(z)}{z} + \frac{1}{z} + \sum_{i=1}^{n}\frac{\bar{N}_i}{z-a_i}\,,
\end{aligned}
\tag{2.20}
$$

where $m(z)$ is the electric meson polynomial. Moreover, at large z we find $\widetilde{T}_{\rm cl} \sim \bar{N}_c/z$ since $\sum_i \bar{N}_i = \bar{N}_c - n$. In the electric pseudoconfining phase, the magnetic singlets vanish classically, thus we find that in the classical chiral ring $\widetilde{m}(z) = 0$.

The low energy theory described by (2.15) can be studied in two steps, following the KSS procedure. First, the $n \times n$ block (2.16) higgses the theory down to

$$
\mathrm{U}(\bar{N}_c) \to \mathrm{U}(\bar{N}_c - n)\,,
$$

and note that $\bar{N}_c - n = n(N_f - 1) - N_c$ as expected from the electric theory, where we integrated out the last massive flavor. At this stage, $\widetilde{q}_\alpha^{\,N_f}$, $q^\alpha_{N_f}$, Y^α_m, Y^s_α for $\alpha = n+1, \ldots, \bar{N}_c$ and $m = 2, \ldots, n$, $s = 1, \ldots, n-1$ conspire to join n massive vector superfields in the fundamental representation of the low energy gauge group $\mathrm{U}(\bar{N}_c - n)$ with mass squared b_1^2. But then as we decompose the adjoint we find that the higgsed flavor gets replaced by a new flavor Y_1^α, Y_α^n for $\alpha = n+1, \ldots, \bar{N}_c$, so the number of flavors does not decrease here. Secondly, the superpotential for the adjoint generates a mass term for this new flavor. Only the leading term $\mathrm{Tr}\,Y^{n+1}$ contributes

$$
\frac{t_n}{n+1}\,\mathrm{Tr}\,Y^{n+1} = t_n Y^\gamma_\alpha \langle Y^{n-1}\rangle^\alpha_\beta Y^\beta_\gamma = t_n b_1^{n-1} Y_1^\gamma Y_\gamma^n\,. \tag{2.21}
$$

The number of flavors effectively decreases by one unit also in the magnetic theory. The singlets $(P_j)^i_{N_f}$ and $(P_j)^{N_f}_i$, $i = 1, \ldots, N_f$ become also massive. Now we can set the massive fields to the solution of their equations of motion and integrate them out. The effective superpotential at a scale below b_1 is $\mathrm{Tr}\,V(\widehat{Y}) + \widehat{\widetilde{q}}\widetilde{m}(\widehat{Y})\widehat{q}$, where the hatted fields transform in the representation of the low energy gauge group $\mathrm{U}(\bar{N}_c - n)$ and we are left with $N_f - 1$ flavors. The matching of the scales goes as follows

$$
\widetilde{\Lambda}^{2\bar{N}_c - N_f}_{\bar{N}_c, N_f} = \frac{m_1 \mu^2}{t_n^2}\widetilde{\Lambda}^{2(\bar{N}_c - n)-(N_f-1)}_{\bar{N}_c - n, N_f - 1}\,. \tag{2.22}
$$

We can use the relation between the scales (1.6) and the electric scale matching and find

$$
\Lambda^{2N_c-(N_f-1)}_{N_c, N_f-1}\widetilde{\Lambda}^{2(\bar{N}_c-n)-(N_f-1)}_{\bar{N}_c-n, N_f-1} = \left(\frac{\mu^2}{t_n^2}\right)^{N_f-1}\,. \tag{2.23}
$$

If we keep flowing to energies below the a_i of (2.17) we will find the usual product of the magnetic theories dual to each electric SQCD block.

This solution can be generalized to the case in which the electric meson polynomial has nonvanishing entries on different flavors. If also the one but last flavor appears in

JHEP11(2004)020

the electric superpotential, i.e. $m(z)^{\tilde{f}}_{f} = m(z)^{N_f}_{N_f} + p(z)^{N_f-1}_{N_f-1}$ with $\deg m(z)^{N_f}_{N_f} = l$ and $\deg p(z)^{N_f-1}_{N_f-1} = l'$, then the new flavor contributes an additional $n \times n$ higgsed block

$$Y = \mathrm{diag}(Y_{N_f}, Y_{N_f-1}, Y_{a_1}, \ldots, Y_{a_n})\,, \tag{2.24}$$

where the first block is always (2.15) and the second block is similar but with the substitution $b_1 \to b_1'$. For what concerns the magnetic quarks, in addition to (2.19) also the one but last flavor is higgsed as follows

$$\begin{aligned} |\tilde{q}^{N_f-1}\rangle &= b_1'|n+1\rangle\,, \\ |q_{N_f-1}\rangle &= \sum_{i=1}^{l} \frac{p_i}{p_1}(b_1')^i|2n+1-i\rangle\,. \end{aligned} \tag{2.25}$$

The magnetic gauge group is now higgsed down to $\mathrm{U}(\bar{N}_c - 2n)$ and in each low energy Seiberg block we have the correspondence $\bar{N}_i = N_f - N_i - 2$.

This classical analysis can be pushed further until we hit the following bound on the number of massless quarks

$$N_f \geq \frac{N_c}{n}\,. \tag{2.26}$$

Suppose in fact that in the meson polynomial $m(z)$ there appear $N_f - N_c/n$ flavors so that we saturate the bound (2.26). Then the magnetic gauge group would be completely higgsed and we will see no low energy SQCD blocks. The solution to this problem is that as the magnetic gauge group is completely higgsed, a new superpotential is triggered by instantons in the broken gauge group and the singlet equations of motion get modified.

2.2 The electric stability bound and magnetic instantons

Let us briefly describe the stability bound on the electric theory [8]. We have seen that a superpotential (2.1) drives the theory to a product of low energy decoupled SQCD, breaking the gauge group down to $\prod_{i=1}^{n} \mathrm{U}(N_i)$. Consider each $\mathrm{U}(N_i)$ SQCD block separately: it is well known that this gauge theory admits a stable vacuum iff the number of flavors is larger than the number of colors [12], i.e. $N_f \geq N_i\ \forall i$. Therefore the original theory admits a stable vacuum iff the bound (2.26) is satisfied.[8]

When we completely break the magnetic gauge group, the weak coupling analysis we carried out is no longer valid due to the presence of instantons. A well known example is $\mathrm{SU}(N_c)$ SQCD with $N_f = N_c + 2$ flavors and its magnetic dual with gauge group $\mathrm{SU}(N_f - N_c) = \mathrm{SU}(2)$ [1]. If we add a mass term for the last electric flavor, the magnetic gauge group gets completely higgsed, so that instantons in the broken $\mathrm{SU}(2)$ generate a superpotential term. By passing to the electric variables, one can see that the sum of the magnetic tree level and instanton superpotentials reproduces the usual nonperturbative superpotential of SQCD with $N_f = N_c + 1$.

[8] The theory is always stable if all the flavors are massive, which is the case we will consider when solving the quantum theory.

We would like to generalize this issue to our case of adjoint SQCD and check whether we can generate an instanton term in the magnetic superpotential when approaching the stability bound. We consider the case in which $\bar{N}_c = n+1$, i.e. we have $N_f = N_c/n+1+1/n$ flavors. At this point, we are just above the bound (2.26) and our classical analysis still makes sense. We further specialize to $n = 2$ and take the electric deformation to be just $t_2 \operatorname{Tr} X^3$.[9] Note that we do not include a mass term for the adjoint. We further add a mass term for the last flavor. Our electric superpotential reads

$$W_{el} = \frac{t_2}{3} \operatorname{Tr} X^3 + m\widetilde{Q}_{N_f} Q^{N_f} . \tag{2.27}$$

The magnetic theory is a U(3) gauge theory defined by the superpotential

$$W_{\rm mag} = -\frac{t_2}{3} \operatorname{Tr} Y^3 + \frac{t_2}{\mu^2}(M_1\widetilde{q}Yq + M_2\widetilde{q}q) + m(M_1)^{N_f}_{N_f} . \tag{2.28}$$

The classical solution (2.15) still applies and the magnetic gauge group is higgsed down to U(1). Since the low energy dynamics is abelian, we may expect instanton effects in the broken gauge group. One can perform a standard analysis of the zero modes in the instanton 't Hooft vertex $\widetilde{\Lambda}^{2\bar{N}_c-N_f}\lambda^{2\bar{N}_c}\psi_Y^{2\bar{N}_c}\psi_{\widetilde{q}}^{N_f}\psi_q^{N_f}$, where ψ_Φ denotes the second component of the chiral superfield Φ. By using the interactions in the tree level action, such as the scalar-fermion-gaugino D-term vertex as well as the superpotential couplings in (2.28), one can read out from this vertex the following contribution to the superpotential

$$W_{\rm inst} = \frac{t_2^{2N_f+3}}{m^2\mu^{4N_f}}(\Lambda^{6-N_f})^2 \det \widehat{M}_2(\widehat{M}_1 \operatorname{cof} \widehat{M}_2) , \tag{2.29}$$

where $\operatorname{cof} M \equiv M^{-1} \det M$ and the hatted fields transform in the $SU(N_f-1)$ low energy flavor symmetry group [13]. Note that this is the contribution by a two-instanton. As explained in [14], this is due to the absence of the mass coupling for the adjoint, that would have been an overall factor in the one-instanton term.

We see that, when hitting the bound (2.26), the classical solution (2.15) is no longer valid, due to the presence of the instanton term that couples the singlets. We can also translate this superpotential to the electric variables by using the scale matching relation (1.6) and the electric low energy scale $\Lambda_{\rm low}^{2N_c-(N_f-1)} = m\Lambda^{2N_c-N_f}$, obtaining the electric superpotential

$$W_{\rm nonpert} = \frac{\det P_2(P_1 \operatorname{cof} P_2)}{t_2^{2N_f}(\Lambda^{2N_c-N_f})^2} , \tag{2.30}$$

In this expression we dropped the hats and the subscript on the scale. It is to be understood as the superpotential of a theory with N_c colors and $N_f = (N_c+1)/2$ flavors. The magnetic instanton superpotential is seen on the electric side as a nonperturbative superpotential arising from strong coupling effects [13], in a very similar way to ordinary SQCD with $N_f = N_c + 1$ flavors.

[9] We consider the $\bar{N}_c = n+1$ rather than the $\bar{N}_c = n$ case because in the latter the magnetic deformation $\operatorname{Tr} Y^{n+1}$ is trivial in the classical chiral ring and the analysis of the instantons is more involved due the presence of additional flat directions.

2.3 The chiral ring

In this section we will find the chiral ring of the quantum theory by solving the generalized Konishi anomaly equations [4, 11]. In appendix A we quote the results we need about DV to set the notations, while for a basic review and a guide to the vast literature we refer to [15]. As a first step, we will consider the case in which the meson superpotential is just a mass term for all the flavors, with no Yukawa-type interactions between quarks and adjoint. While the solution of the electric theory is standard, the anomaly equations in the magnetic theory are somewhat different, due to the presence of the gauge singlets. This massive case is useful to illustrate the general procedure without worrying about the rich analytic structure of the generators of the chiral ring, that we will encounter later.

The electric theory. We will focus on the case in which the electric meson superpotential is just a mass term

$$W_{el} = \mathrm{Tr}\ V(X) + \widetilde{Q}_{\tilde{f}} m_f^{\tilde{f}} Q^f \,, \tag{2.31}$$

where m is a diagonal matrix. If the second derivatives of $V(z)$ at the saddle points are nonvanishing, all the fields will be massive and it makes sense to use the effective action as a function of the glueball superfield S. We will be interested in the chiral operators (2.3).

The solution of the anomaly equation for the resolvent $R(z)$ gives

$$2R(z) = V'(z) - \sqrt{V'(z)^2 + f(z)} \,, \tag{2.32}$$

where $f(z)$ is a $n-1$ degree polynomial $f(z) = f_1 + \cdots + f_n z^{n-1}$. This defines the curve of the electric theory to be the hyperelliptic Riemann surface $y^2 = V'(z)^2 + f(z)$.

Since the meson polynomial $m(z)$ is just constant, the anomaly equation for the matrix $M(z)$ reduces to the following simple form $[M(z)m]_- = R(z)$, where we suppressed flavor indices. The solution is

$$M(z) = R(z) m^{-1} \,, \tag{2.33}$$

M being a diagonal matrix.

The anomaly equation for $T(z)$ is

$$[y(z)T(z)]_- + [\mathrm{tr}\ m'(z)M(z)]_- = 0 \,, \tag{2.34}$$

but since $m(z) = \mathrm{const}$ the last term drops. The solution is

$$T(z) = \frac{c(z)}{\sqrt{V'(z)^2 + f(z)}} \,, \tag{2.35}$$

where $c(z)$ is another $n-1$ degree polynomial $c(z) = c_1 + \cdots + c_n z^{n-1}$. Since m is not z-dependent, in the electric theory the fundamentals do not influence the solution for T .

The parameters f_j, c_j are related to the glueballs S_i of the low energy SQCD blocks and the ranks N_i of their gauge groups as follows

$$\begin{aligned} S_i &= \oint_{A_i} R(z) dz \,, \\ N_i &= \oint_{A_i} T(z) dz \,, \end{aligned} \tag{2.36}$$

where A_i is classically a contour around a_i. At the quantum level, each stationary point a_i opens up into a branch cut for $R(z)$ and the contour A_i actually encircles the two branch points. One can get exact formulae for the total glueball $S = \sum_i S_i$ and the rank of the high energy gauge group $N_c = \sum_i N_i$ by looking at the $1/z$ terms in (2.32) and (2.35), since choosing a contour A around all the branch points is equivalent to closing it around ∞. In this way we can fix the first coefficient of the polynomials $c(z)$ and $f(z)$

$$S = -\frac{f_n}{4t_n}, \qquad N_c = \frac{c_n}{t_n}\,. \tag{2.37}$$

We calculate now the relevant relations in the chiral ring. We can extract from (2.33) the mesons operators by

$$\widetilde{Q}X^{j-1}Q = \oint_A z^{j-1}M(z)\,, \tag{2.38}$$

where the contour A encircles all the branch points of the resolvent $R(z)$, obtaining

$$\widetilde{Q}X^{j-1}Q = -\sum_{i=1}^{n}\frac{a_i^{j-1}f(a_i)}{4mV''(a_i)}\,, \tag{2.39}$$

for $j = 1,\ldots,n$, coming from the negative power expansion of the first term in the semi-classical expansion of the resolvent

$$R(z) = -\frac{f(z)}{4V'(z)} + \mathcal{O}\left(\frac{f(z)^2}{V'(z)^3}\right), \tag{2.40}$$

In particular we find the usual Konishi anomaly

$$\widetilde{Q}_f Q^f = \frac{N_f S}{m}\,, \tag{2.41}$$

where we used (2.37). Higher meson operators receive additional contributions from the semiclassical expansion. The single trace of the adjoint X can be obtained as the coefficients of inverse powers of z in the expansion of $T(z)$ at large z

$$\mathrm{Tr}\,X^j = \oint_A z^j T(z)dz\,, \tag{2.42}$$

where A circles all the the branch points of the resolvent $R(z)$. Expanding $T(z)$ we get

$$T(z) = \frac{c(z)}{V'(z)} - \frac{c_n f_n}{2t_n^3 z^{n+2}} + \cdots \tag{2.43}$$

and we can extract the chiral operators

$$\mathrm{Tr}\,X^j = \sum_{i=1}^{n}\frac{c(a_i)a_i^j}{V''(a_i)} + \delta^j_{n+1}\frac{2N_c S}{t_n}\,, \tag{2.44}$$

for $j = 1,\ldots,n+1$. Clearly the equation $V'(X) = 0$ is obeyed in the chiral ring, but relations obtained by multiplying it with X get quantum corrections.

The magnetic theory. The magnetic theory corresponding to (2.31) has a tree level superpotential

$$W_{\rm mag} = {\rm Tr}\,\bar{V}(Y) + \tilde{q}^{\tilde{f}}\tilde{m}^{f}_{\tilde{f}}(Y)q_f + \bar{m}\,{\rm tr}(P_1)\,. \qquad (2.45)$$

Note that the quantities appearing in (2.45) are the magnetic ones, as explained at the beginning of this section. In particular we have that

$$\tilde{m}(z) = -\frac{1}{\mu^2}\oint_{\tilde{A}} \frac{\bar{V}'(\zeta)-\bar{V}'(z)}{\zeta - z}P(\zeta)\,, \qquad (2.46)$$

and, inverting this, we find the gauge singlets

$$P(z) = -\mu^2\left[\frac{\tilde{m}(z)}{\bar{V}(z)}\right]_{-n}\,, \qquad (2.47)$$

We are ready to use now the anomaly equations. The form of $\tilde{R}(z)$, which is independent on the fundamentals, will be the same as for the electric theory

$$2\tilde{R}(z) = \bar{V}'(z) - \sqrt{\bar{V}(z)^2 + \bar{f}(z)}\,, \qquad (2.48)$$

where the quantum deformation $\bar{f}(z)$ is a degree $n-1$ polynomial. Since we will see that the quantum deformations on both sides are equivalent under the offshell duality map, we will conclude that the magnetic theory has the same curve of the electric one.

In addition to the usual anomaly equations, that we encountered in the electric theory, there are new ones following from variations of the gauge singlets P's. Since P is not coupled to the gauge fields, these are just its equations of motion. For the special case we are studying, after rearranging the equations, (2.10) reduce to

$$\begin{aligned} \bar{t}_n\tilde{q}Y^{n-1}q &= \bar{m}\mu^2\,, \\ \tilde{q}Y^{j-1}q &= 0\,, \qquad j = 1,\dots,n-1\,. \end{aligned} \qquad (2.49)$$

On the other hand, the role of the electric meson polynomial is played now by $\tilde{m}(z)$. The anomaly equation for the meson generator is then

$$[\tilde{M}(z)\tilde{m}(z)]_- = \tilde{R}(z)\,. \qquad (2.50)$$

Its generic solution is

$$\tilde{M}(z) = \tilde{R}(z)\tilde{m}^{-1}(z) + r(z)\tilde{m}^{-1}(z)\,, \qquad (2.51)$$

in our case all the matrices being diagonal. The crucial piece of information about the magnetic theory is the quantum expression of $\tilde{m}(z)$, which contains the gauge singlets and fixes the analytic properties of the meson generator. The way in which (2.51) is supposed to be used is the following

1. We fix the polynomial $r(z)$ such that there are no additional singularities in (2.51) arriving from the zeroes of $\tilde{m}(z)$.

2. We fix the polynomial $\tilde{m}(z)$ imposing that the mesons $\tilde{q}X^{j-1}q$ extracted from (2.51) fulfill the singlet equations of motion (2.49). In this way we fix also $P(z)$.

JHEP11(2004)020

The unique solution to these requirements is

$$r(z) = 0\,, \tag{2.52}$$

and

$$\widetilde{m}(z) = -\frac{\overline{f}(z)}{4\bar{m}\mu^2}\,. \tag{2.53}$$

Since $\widetilde{m}(z)$ is proportional to $\overline{f}(z)$ and the the resolvent $\widetilde{R}(z)$ vanishes at the zeroes of $\overline{f}(z)$, we see that $\widetilde{m}^{-1}(z)$ does not give additional singularities in (2.51). The analytic structure of $\widetilde{M}(z)$ in this case turns out to be very simple, while the singlets are

$$P(z) = \frac{1}{4\bar{m}}\left[\frac{\overline{f}(z)}{\bar{V}\,'(z)}\right]_{-n}\,, \tag{2.54}$$

where the expansion in inverse powers of z is understood to stop at z^{-n}. Comparing with (2.39) we see that the matching

$$P_j = \widetilde{Q}X^{j-1}Q\,, \tag{2.55}$$

for $j = 1, \ldots, n$, is implied for a sign choice which will be discussed later. Of course we could go backwards and requiring (2.55) prove the form of the Kutasov kernel $\frac{\bar{V}\,'(\zeta)-\bar{V}\,'(z)}{\zeta - z}$ which determines the form of the fundamental magnetic superpotential. We can extract the expectation values of the magnetic singlets out of (2.54)

$$P_j = \frac{1}{4\bar{m}}\sum_{i=1}^{n}\frac{\bar{f}(\bar{a}_i)\bar{a}_i^{j-1}}{\bar{V}\,''(\bar{a}_i)} = -\sum_{i=1}^{n}\frac{\bar{a}_i^{j-1}\bar{S}_i}{\bar{m}}\,, \tag{2.56}$$

where we used the definition of the glueballs in (2.36).

We can now calculate $\widetilde{T}(z)$. Its anomaly equation is

$$\left[\widetilde{y}(z)\widetilde{T}(z)\right]_- + \mathrm{tr}\left[\widetilde{m}'(z)\widetilde{M}(z)\right]_- = 0\,. \tag{2.57}$$

The solution here, as opposed to (2.35), depends also on the fundamentals

$$\widetilde{T}(z) = \frac{1}{\widetilde{y}(z)}\left[-\widetilde{m}'(z)\widetilde{M}(z) + \overline{c}(z)\right] \tag{2.58}$$

with $\widetilde{m}'(z)$, $\widetilde{M}(z)$ given by (2.54) and (2.51). Since $\overline{c}(z)$ is a polynomial of degree $n-1$, while $\widetilde{m}'(z)\widetilde{M}(z)$ starts with z^{-2}, the contribution of the fundamentals will start only from the power z^{-n-2}. Recalling that $\widetilde{m}$ is a diagonal matrix, we expand (2.58) at large z

$$\widetilde{T}(z) = \frac{\overline{c}(z)}{\bar{V}\,'(z)} - \frac{\overline{c}_n\bar{f}_n}{2\bar{t}_n^2 z^{n+2}} + N_f\frac{\bar{f}_n}{4\bar{t}_n^2}\frac{1}{z^{n+1}}\frac{\bar{f}'(z)}{\bar{f}(z)}\,, \tag{2.59}$$

and the chiral ring is

$$\mathrm{Tr}\,Y^j = \sum_{i=1}^{n}\frac{\bar{a}_i^j\overline{c}(\bar{a}_i)}{\bar{V}\,''(\bar{a}_i)}\,, \tag{2.60}$$

for $j = 0, \ldots, n$. The first operator which will receive a contribution from the last two terms in (2.59) will be $\mathrm{Tr}\, Y^{n+1}$ which in the magnetic theory becomes[10]

$$\mathrm{Tr}\, Y^{n+1} = \sum_{i=1}^{n} \frac{\bar{c}(\bar{a}_i)\bar{a}_i^{n+1}}{\bar{V}''(\bar{a}_i)} + \frac{2\bar{N}_c\bar{S}}{\bar{t}_n} - N_f \frac{\bar{S}(n-1)}{\bar{t}_n}\,. \tag{2.61}$$

We would like to stress again a basic property of the solution (2.53). Since it is proportional to $\bar{f}(z)$, the meson generator $\widetilde{M}(z)$ and also $\widetilde{T}(z)$ have a very simple analytic structure in both sheets, as opposed to the generic cases we will solve below. Because of this fact, we will be able to see that the electric-magnetic duality map here works exactly offshell.

2.4 The effective actions

Once that we have solved the chiral ring, we can determine the superpotential part of the low energy effective action by integrating the derivatives with respect to the parameters appearing in the lagrangian, which are the expectation values of the chiral operators we just computed above. This offshell effective action will be valid at energies above the glueball mass, that sets the scale of mass gap.

The electric theory. The electric couplings are m and $t_1, \ldots, t_n$. It is covenient to use as independent parameters t_n and $\widehat{t_j} = \frac{t_j}{t_n}$ for $j = 1, \ldots, n-1$. The parameters $\widehat{t_j}$ are homogenous polynomials in a_i. The derivatives of the effective action are

$$\frac{\partial W_{\text{eff}}}{\partial \widehat{t_j}} = t_n \frac{1}{j+1} \langle \mathrm{Tr}\, X^{j+1} \rangle = -\frac{1}{j+1} \sum_{i=1}^{n} \frac{t_n c(a_i) a_i^{j+1}}{4V''(a_i)}\,, \tag{2.62}$$

for $j = 1, \ldots, n-1$ and

$$\frac{\partial W_{\text{eff}}}{\partial t_n} = \frac{1}{n+1} \langle \mathrm{Tr}\, X^{n+1} \rangle = -\frac{1}{n+1} \sum_{i=1}^{n} \frac{c(a_i) a_i^{n+1}}{4V''(a_i)} + \frac{1}{n+1} \frac{2N_c S}{t_n}, \tag{2.63}$$

$$\langle \widetilde{Q}_f Q^f \rangle = \frac{\partial W_{\text{eff}}}{\partial m} = \frac{N_f S}{m}\,. \tag{2.64}$$

Since we are looking for the offshell effective action, these equations are supposed to be integrated at fixed S_i, N_i. Now observe that (2.64) and the second term in (2.63) satisfy the integrability condition by themselves. Therefore we can integrate them separately and there is a solution $\mathcal{W}_{\text{eff}}$ without them. The general effective action we obtain by (2.62), (2.63) and (2.64) is

$$W_{\text{eff}} = \mathcal{W}_{\text{eff}} + \frac{2N_c S}{n+1} \log t_n + N_f S \log m + [t_j, m - \text{independent terms}]\,. \tag{2.65}$$

Let us consider the coupling independent terms. There are two contributions, the first is the one-loop exact renormalization of the gauge field kinetic term $(2N_c - N_f) S \log \Lambda$, that contains the dynamically generated scale Λ through the running gauge coupling constant.

[10]We used the fact that $\oint f'/f = \#$ zeroes of f.

Then we have a Veneziano-Yankielowicz type superpotential $bS(\log S - 1)$. One can fix the numerical coefficient b by requiring that the effective action is $\mathrm{U}(1)_R$ invariant. Since the R-current and the dilatation current lie in the same $\mathcal{N} = 1$ supermultiplet, this is the same as fixing them by dimensional analysis. By the usual localization trick, we promote the couplings to background chiral superfields so that we can assign them a charge. The dimensions Δ of the various fields are

	Δ
S	3
t_j	$2-j$
m	1
$\Lambda^{2N_c-N_f}$	$2N_c - N_f$

(2.66)

so that we find $b = -2N_c/(n+1)$. Since $\mathcal{W}_{\text{eff}}$ is invariant by itself we get the effective superpotential

$$W_{\text{eff}} = \mathcal{W}_{\text{eff}} + S \log \frac{\Lambda^{2N_c-N_f} t_n^{\frac{2N_c}{n+1}} m^{N_f}}{S^{\frac{2N_c}{n+1}}} + \frac{2N_c}{n+1} S \,. \tag{2.67}$$

We will now turn to the evaluation of the term $\mathcal{W}_{\text{eff}}$. It is most convenient to parameterize the degree $n-1$ polynomial $c(z)$ in the following way

$$c(z) = V'(z) \sum_{i=1}^{n} \frac{h_i}{z - a_i} \,. \tag{2.68}$$

where $N_c = \sum_{i=1}^n h_i$. The n coefficients h_i are fixed by the contour integral

$$h_i = \oint_{A_i} \frac{c(z)}{V'(z)} \,, \tag{2.69}$$

so that classically we have just $h_i = N_i$. Using this parametrization we can rewrite the relevant part of (2.62) and (2.63) as

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial t_j} = \sum_{i=1}^{n} h_i a_i^j \,. \tag{2.70}$$

In particular, we see that $\operatorname{Tr} X^j = \sum_i h_i a_i^j$ for $j = 1, \dots, n$, while $\operatorname{Tr} X^{n+1}$ contains in addition the last term in (2.44). The coefficients h_i depend on t_j, S_i and N_k, as we can see from (2.69). It is convenient to use in place of the glueballs S_i the new variable

$$y = \sum_{i=1}^{n} \log S_i \,, \tag{2.71}$$

and $n-1$ independent ratios of glueballs, e.g. $\frac{S_1}{S_n}, \dots, \frac{S_{n-1}}{S_n}$. Introduce now the following functions

$$d_i = h_i - e^y \int_{-\infty}^{y} dy' e^{-y'} \frac{\partial h_i}{\partial y'} \,. \tag{2.72}$$

We claim that integrating (2.70) we find

$$\mathcal{W}_{\text{eff}} = \sum_{i=1}^{n} d_i V(a_i)\,. \tag{2.73}$$

In the appendix B we prove, up to an assumption of integrability, that indeed differentiating (2.73) one recovers (2.70).

Putting everything together, the effective superpotential of the electric theory is

$$W_{\text{eff}} = \sum_{i=1}^{n} d_i V(a_i) + S \log \frac{\Lambda^{2N_c - N_f} t_n^{\frac{2N_c}{n+1}} m^{N_f}}{S^{\frac{2N_c}{n+1}}} + \frac{2N_c}{n+1} S\,. \tag{2.74}$$

The magnetic theory. We can follow again the same procedure of integrating the expectation values with respect to the parameters, but now we have a new coupling μ and the derivative with respect to $\bar{t}_j$ gets a contribution also from the magnetic fundamentals and singlets

$$\frac{\partial \overline{W}_{\text{eff}}}{\partial \bar{t}_j} = \frac{1}{j+1} \langle \operatorname{Tr} Y^{j+1} \rangle + \frac{1}{\mu^2} \sum_{i=1}^{j} \langle P_i \widetilde{q} Y^{j-i} q \rangle\,. \tag{2.75}$$

for $j = 1, \ldots, n$. Then we have

$$\mu^2 \frac{\partial \overline{W}_{\text{eff}}}{\partial \mu^2} = \bar{m} \langle \operatorname{tr} P_1 \rangle, \qquad \frac{\partial \overline{W}_{\text{eff}}}{\partial \bar{m}} = \langle \operatorname{tr} P_1 \rangle\,, \tag{2.76}$$

where we used the fact that the expectation values of gauge invariant chiral operators factorize and the singlet equations of motion (2.10), which are exact in the quantum theory. We can substitute the expectation values (2.56), (2.60), and (2.61) into (2.75), obtaining

$$\frac{\partial \overline{W}_{\text{eff}}}{\partial \bar{t}_j} = \frac{1}{j+1} \sum_{i=1}^{n} \frac{\bar{c}(\bar{a}_i) \bar{a}_i^{j+1}}{\bar{V}''(\bar{a}_i)} + \frac{\delta_n^j}{n+1} \frac{2(\bar{N}_c + N_f)\bar{S}}{\bar{t}_n}\,, \tag{2.77}$$

for $j = 1, \ldots, n$ and into (2.76)

$$\mu^2 \frac{\partial \overline{W}_{\text{eff}}}{\partial \mu^2} = -N_f S, \qquad \frac{\partial \overline{W}_{\text{eff}}}{\partial \bar{m}} = -N_f \frac{S}{\bar{m}}\,. \tag{2.78}$$

The first term in (2.77) is analogous to the corresponding electric one in (2.62). We assume, as in that case, that it satisfies the integrability condition by itself and integrate it to obtain $\overline{\mathcal{W}}_{\text{eff}}$. This is formally equal to (2.73) but with magnetic quantities instead. On the other hand, also (2.78) and the second term in (2.77) satisfy the integrability condition, so that we find

$$\overline{W}_{\text{eff}} = \overline{\mathcal{W}}_{\text{eff}} + \frac{2(\bar{N}_c + N_f)}{n+1} \bar{S} \log \bar{t}_n - 2N_f \bar{S} \log \mu - N_f \bar{S} \log \bar{m} + [\bar{t}_j, \bar{m}, \mu - \text{indep. terms}]\,. \tag{2.79}$$

Then we need to add the magnetic one-loop renormalization of the gauge fields $(2\bar{N}_c - N_f)\bar{S} \log \widetilde{\Lambda}$ and the Veneziano-Yankielowicz type superpotential $\bar{b}\bar{S}(\log \bar{S} - 1)$. Again we fix the coefficient $\bar{b}$ requiring $\mathrm{U}(1)_R$ invariance, as we did for the electric case, and get

$\bar b = 2(nN_f - \bar N_c)/(n+1)$. Putting everything together we obtain the magnetic effective action

$$\overline{W}_{\rm eff} = \sum_{i=1}^{n} \bar d_i \bar V(\bar a_i) + \bar S \log \frac{\widetilde{\Lambda}^{2\bar N_c - N_f} \bar t_n^{\frac{2(\bar N_c + N_f)}{n+1}} \bar S^{\frac{2(nN_f - \bar N_c)}{n+1}}}{\bar m^{N_f} \mu^{2N_f}} - \frac{2(nN_f - \bar N_c)}{n+1} \bar S \,. \qquad (2.80)$$

2.5 The offshell duality map

At this point we will look for the duality map between the electric and magnetic operators in the chiral ring. As we discussed in the introduction, in this case the duality holds exactly offshell. First we will consider the match of the meson operators and then the effective actions.

The gauge singlets equations of motion (2.10) are exact in the chiral ring of the magnetic quantum theory. They tell us that the magnetic meson operators $\widetilde{q} Y^{j-1} q$ are trivial. They are replaced by the gauge singlets, which represent the electric mesons through a Legendre transform, as it is clear from the expression of $\widetilde{m}(z)$ in the magnetic tree level superpotential (2.5). Therefore we should match directly the electric mesons with the corresponding magnetic gauge singlets through the relation

$$P_j = \widetilde{Q} X^{j-1} Q \,, \qquad (2.81)$$

independently on the other relations between the gauge groups. Comparing the two expressions (2.56) and (2.39)

$$\widetilde{Q} X^{j-1} Q = \sum_{i=1}^{n} \frac{a_i^{j-1} S_i}{m}, \qquad P_j = - \sum_{i=1}^{n} \frac{\bar a_i^{j-1} \bar S_i}{\bar m},$$

for $j = 1, \ldots, n$, we get the relations

$$S_i = -\bar S_i \,, \qquad m = \bar m \,, \qquad (2.82)$$

while the roots of the electric and magnetic polynomials for the adjoint coincide $a_i = \bar a_i$, i.e. the electric polynomial $V'(z)$ and the magnetic one $\overline{V}'(z)$ are identified up to a minus sign. Let us recall the definition (2.36) of the glueballs in terms of the resolvent

$$S_i = - \frac{f(a_i)}{4V''(a_i)} \,,$$

which holds both for the electric and magnetic theories with the respective quantities. The relation (2.82) then fixes the the duality map as

$$f(z) = \bar f(z), \qquad V'(z) = -\overline{V}'(z) \,. \qquad (2.83)$$

This last relation, in particular, tells us that electric and magnetic theories have the same curve

$$y^2 = V'(z)^2 + f(z) \,. \qquad (2.84)$$

Now let us consider the electric and the magnetic effective actions (2.74) and (2.80). By comparing their second and third terms we get again the match between the glueballs

JHEP11(2004)020

and the mass terms (2.82) and $t_n = -\bar{t}_n$, which fixes the ambiguity in the sign choice of (2.83), together with the scale matching relation

$$\Lambda^{2N_c - N_f} \widetilde{\Lambda}^{2\bar{N}_c - N_f} = t_n^{-2N_f} \mu^{2N_f} \,, \tag{2.85}$$

and the usual relation between the electric and magnetic gauge groups $\bar{N}_c = nN_f - N_c$. The scale matching (2.85) is consistent with the fact that $\log \Lambda^{2N_c - N_f}$ and $\log \widetilde{\Lambda}^{2\bar{N}_c - N_f}$ are the sources for the respective electric and magnetic total glueballs and that we found $\bar{S} = -S$. Let us analyze in more detail the relation between the gauge groups. The rank of the electric and magnetic gauge groups fixes the pole at infinity of $T(z)$

$$N_c = \oint_A T(z) \,, \tag{2.86}$$

where A is the large contour, and analogously for the magnetic theory. The matching $\bar{N}_c = nN_f - N_c$ translates into the following relation

$$\oint_{\widetilde{A}} \widetilde{T}(z) = N_f \oint_A \frac{V'(z)}{V''(z)} - \oint_A T(z) \,. \tag{2.87}$$

We evaluate the contour integrals by expanding (2.35) and (2.58) at large z and get

$$\overline{c}(z) = c(z) - N_f V''(z) \,. \tag{2.88}$$

By the definition (2.69) of the coefficients h_i we find that $\overline{h}_i = N_f - h_i$ or equivalently $\overline{d}_i = N_f - d_i$, which fixes the map between the operators

$$\mathrm{Tr}\, Y^j = -\,\mathrm{Tr}\, X^j + N_f \sum_{i=1}^{n} a_i^j \,, \tag{2.89}$$

for $j = 1, \ldots, n$, in agreement with the KSS results [9]. The match between the electric and magnetic $\mathcal{W}_{\rm eff}$ using the relation $\bar{d}_i = N_f - d_i$ shows that the magnetic effective superpotential contains an additional Y-independent term, which in this case is just $N_f \sum_i V(a_i)$.

The classical limit of the coefficients h_i is N_i, the rank of each low energy SQCD block. Thus we recover the usual matching relation $\bar{N}_i = N_f - N_i$, which is somewhat different from the one we found in our classical analysis of (2.18), which anyway was only valid in the case where the number of massive electric quarks is less than $N_f - N_c/n$, because of the stability bound. The higgsed blocks in the magnetic adjoint we found in (2.14) and (2.24), that were responsible for the singularities in (2.20), are an artifact of the classical theory. When we pass to the full quantum theory, in this pseudoconfining case all the classical singularities of $\widetilde{M}(z)$ are smoothed out, and this is the reason why we get back the usual Seiberg duality map $\bar{N}_i = N_f - N_i$ for the rank of the gauge groups of the low energy SQCD blocks.

2.6 The generic pseudoconfining case

In the last section we saw that, in the case of massive quarks without Yukawa couplings, duality works offshell, that is at the level of the dynamical effective actions. We will consider

JHEP11(2004)020

in this section the most generic pseudoconfining case, where in addition to the mass terms for the quarks we allow for a generic z-dependent meson polynomial. As a consequence, the analytic properties of the various resolvents in the quantum chiral ring get more involved and in the end the match between electric and magnetic quantities will not hold anymore exactly offshell, but we expect it to hold only onshell. We will not compute the effective action, as we did above, but we will match the electric mesons with the magnetic singlets and find a map that reproduces the Konishi anomaly in each low energy SQCD block as a classical equation in the magnetic theory.

The electric theory. Let us consider the electric theory with a generic yet diagonal meson polynomial

$$\begin{aligned} W_{el} &= \operatorname{Tr} V(X) + \widetilde{Q}^{\tilde{f}} m(X)_{\tilde{f}}^{f} Q_f\,, \\ m(z)_{f}^{\tilde{f}} &= \sum_{i=1}^{l+1} m_i z^{i-1} \delta_{f}^{\tilde{f}}\,. \end{aligned} \tag{2.90}$$

We denote the roots of $m(z)$ as x_k, for $k = 1, \ldots, l$. The degree of the polynomial $m(z)$ is at most $n-1$ and its constant term m_1 must be nonzero for all the flavors in order for the theory to be massive. The classical pseudoconfining vacuum is (2.4), while the generators (2.3) of the classical chiral ring all vanish except

$$T_{\rm cl}(z) = \sum_{i=1}^{n} \frac{N_i}{z - a_i}\,. \tag{2.91}$$

This phase is characterized by a vanishing classical expection value for the fundamentals.

Let us consider the generalized Konishi anomaly equations. The resolvent $R(z)$ is still given by (2.32). The story is different for $M(z)$, the generator of the mesons. When solving its anomaly equation, we have to cancel the additional singularities coming from the zeroes of $m(z)$. We have to specify the boundary conditions coming from our choice of the vacuum. In this pseudoconfining case, $M(z)$ is regular in the first sheet (up to the residue at infinity). Implementing these boundary conditions we find

$$M(z) = \frac{R(z)}{m(z)} - \sum_{k=1}^{l} \frac{R(q_k)}{z - x_k} \frac{1}{m'(x_k)}\,. \tag{2.92}$$

Let us extract the expectation value of the mesons. We can evaluate (2.38) by expanding semiclassically the resolvent in powers of $f(z)/V'(z)^2$ as in (2.40) and find

$$\widetilde{Q} X^{j-1} Q = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{4 m(a_i) V''(a_i)} + \cdots\,, \tag{2.93}$$

where we showed only the leading approximation. Here we see the crucial difference between the purely massive case (2.39) and this general case. There, we took the semiclassical expansion and then we opened up the contour A to the big circle, throwing away all the higher terms in the expansion. Here, we cannot open up the contour A after taking the semiclassical expansion, because in this process we would hit the additional poles at the zeroes of $m(z)$ for each term in the expansion. Due to the richer analytic structure, we are forced to keep in (2.93) all the semiclassical expansion. We will see that a duality map exists at the first order in this expansion.

A similar story carries on to the last anomaly equation (2.34), whose solution with the classical limit (2.91) is

$$T(z) = \sum_{k=1}^{l} \frac{1}{2(z-x_k)} - \sum_{k=1}^{l} \frac{y(q_k)}{2y(z)(z-x_k)} + \frac{c(z)}{y(z)}\,, \tag{2.94}$$

where

$$c(z) = V'(z) \sum_{i=1}^{n} \frac{h_i}{z-a_i} - \frac{1}{2} \sum_{k=1}^{l} \frac{V'(z)-V'(x_k)}{z-x_k}\,, \tag{2.95}$$

is a degree $n-1$ polynomial whose leading coefficient is $c_n/t_n = N - l/2$. Note that in this case the fundamentals do contribute to $T(z)$. We have considered a convenient parametrization of (2.95) similar to the one in (2.68) but now slightly modified to take into account the more complicated analytic structure. We still have $\sum_i h_i = N_c$. Since the roots x_k of the meson polynomial $m(z)$ are supposed to be very large in the semiclassical limit, we see that the definition of the coefficients h_i is still (2.69), the last term in (2.95) not contributing to the contour integral. Now we can integrate the generator on the contour A to obtain the expectation values. The first term in (2.94) does not contribute because the x_k's lie outside the contour and we obtain

$$\operatorname{Tr} X^j = \sum_{i=1}^{n} h_i a_i^j + \sum_{k=1}^{l} R(x_k) \sum_{i=1}^{n} \frac{a_i^j}{(a_i-x_k)V''(a_i)} + \delta_{n+1}^j \frac{2S(N-l/2)}{t_n} + \cdots\,, \tag{2.96}$$

for $j = 1, \ldots, n+1$. By $\cdots$ we denote higher terms in the semiclassical expansion (2.40).

The magnetic theory and the match. The magnetic theory corresponding to (2.90) has the tree level superpotential

$$W_{\mathrm{mag}} = \operatorname{Tr} \bar{V}(Y) + \widetilde{q}\widetilde{m}(P,Y)q + \oint \bar{m}(z)P(z)\,, \tag{2.97}$$

where we use the same notations as in (2.46) and $\bar{m}(z)$ corresponds to the electric meson polynomial. This phase is characterized by a vanishing classical value of the singlets and thus also of $\widetilde{m}(z)$.

Let us solve the anomaly equations. The resolvent $\widetilde{R}(z)$ is still given by (2.48). The anomaly equation for the generator of the magnetic mesons is always (2.50), whose general solution is

$$\widetilde{M}(z) = \widetilde{R}(z)\widetilde{m}^{-1}(z) + r(z)\widetilde{m}^{-1}(z)\,. \tag{2.98}$$

We recall that the polynomial $r(z)$ is fixed in order to cancel the additional singularities coming from the zeroes of $\widetilde{m}(z)$. Then $\widetilde{m}(z)$ is fixed by imposing that the magnetic singlet equations of motion are satisfied. Denote the roots of the degree $n-1$ polynomial $\widetilde{m}(z)$ as

e_k, for $k = 1, \dots, n-1$. In this case our boundary conditions are such that $\widetilde{M}(z)$ is regular in the first sheet at the zeroes e_k

$$\frac{r(z)}{\widetilde{m}(z)} = -\sum_{k=1}^{n-1} \frac{\widetilde{R}(e_k)}{z - e_k} \frac{1}{\widetilde{m}'(e_k)} \,. \tag{2.99}$$

Note that in the previous case (2.54) there was no need to keep the polynomial $r(z)$, since $\widetilde{m}(z)$ was proportional to the quantum deformation $\bar{f}(z)$ of the resolvent. In that case, no additional singularity was present. Now the story is quite different and to find the result we should first rewrite the singlet equations of motion in a more convenient way. First note that, just as we can usually trade the glueballs S_i for the coefficients of the quantum deformation $f(z)$ [16], we can also trade the n singlets P_l for the n coefficients of the polynomial $\widetilde{m}(z) = \sum_{l=1}^{n} \widetilde{m}_l z^{l-1}$, that are a linear combination thereof

$$\widetilde{m}_l = -\frac{1}{\mu^2} \sum_{k=l}^{n} \bar{t}_k P_{k-l+1} \,. \tag{2.100}$$

Now we cast the superpotential in a suitable form to replace the $P(z)$ with the $\widetilde{m}(z)$. Recall that the singlets are fixed by $\widetilde{m}(z)$ as in (2.47). By using (2.97), the relevant part of the superpotential we need is

$$\widetilde{q}\widetilde{m}(Y)q - \mu^2 \oint_{\widetilde{A}} \frac{\bar{m}(z)\widetilde{m}(z)}{\bar{V}\,'(z)} \,. \tag{2.101}$$

Differentiating w.r.t. $\widetilde{m}_l$ we get

$$\widetilde{q}Y^{l-1}q - \mu^2 \oint_{\widetilde{A}} z^{l-1} \frac{\bar{m}(z)}{\bar{V}\,'(z)} = 0 \,, \tag{2.102}$$

for $l = 1, \dots, n$, that we can also write as

$$\oint_{\widetilde{A}} z^{l-1} \left[\widetilde{M}(z) - \mu^2 \frac{\bar{m}(z)}{\bar{V}\,'(z)} \right] = 0 \,, \tag{2.103}$$

Note that while in the electric case the zeroes of $m(z)$ are very large in the semiclassical regime, in the magnetic case it turns out that the zeroes of $\widetilde{m}(z)$ do lie inside the $\widetilde{A}$ contour, as we will see explicitly in section 4 for the cubic superpotential. We can expand the resolvent semiclassically as in (2.40) and only will the residue at the zeroes of $V'(z)$ contribute. Remember that the singlet equations of motion (2.103) are supposed to fix the unknown polynomial $\widetilde{m}(z)$. Indeed the solution of (2.103) at the first order in the semiclassical expansion is

$$4\mu^2 \bar{m}(\bar{a}_i)\widetilde{m}(\bar{a}_i) = -\bar{f}(\bar{a}_i) \,, \tag{2.104}$$

for $i = 1, \dots, n$, where $\bar{a}_i$ are the roots of $\bar{V}\,'(z)$ and the flavor indices are suppressed (note that they are not summed over). Eq. (2.104) consists of n conditions that account for the n unknown coefficients $\widetilde{m}_l$.

Some comments are in order. The classical limit of (2.104) is well defined, since both sides vanish (remember that classically the singlets vanish in this phase). Now look at the meson generator $\widetilde{M}(z)$ in (2.98) with boundary conditions (2.99). At the quantum level it is regular in the first semiclassical sheet, while it has $n-1$ poles on the second sheet. Nevertheless, when taking the classical limit, both $\widetilde{R}(z)$ and $\widetilde{m}(z)$ vanish, but the result is a nonvanishing classical value for $\widetilde{M}(z)$, that reproduces our classical understanding of the theory being higgsed, as explained in section 2.1. Here we see again the same issue discussed at the end of section 2.5. The singularities of $\widetilde{M}(z)$ on the first sheet are an artifact of the classical theory in the pseudoconfining case: they are smoothed out in the full quantum theory.

We can complete the analysis of the magnetic chiral ring by solving the anomaly equation (2.57) for $\widetilde{T}(z)$ where $\widetilde{m}(z)$ and $\widetilde{M}(z)$ are given by (2.98) and (2.104) and get

$$\widetilde{T}(z) = \sum_{k=1}^{n-1} \frac{1}{2(z-e_k)} - \sum_{k=1}^{n-1} \frac{\widetilde{y}(e_k)}{2\widetilde{y}(z)(z-e_k)} + \frac{\bar{c}(z)}{\widetilde{y}(z)}\,, \tag{2.105}$$

where we can choose the following parametrization for the degree $n-1$ polynomial $\bar{c}(z)$

$$\bar{c}(z) = \bar{V}'(z) \sum_{i=1}^{n} \frac{\bar{h}_i}{z-\bar{a}_i} - \frac{1}{2} \sum_{k=1}^{n-1} \frac{\bar{V}'(z) - \bar{V}'(e_k)}{z-e_k}\,, \tag{2.106}$$

where we can fix $\bar{c}_n/\bar{t}_n = \bar{N}_c - n + 1$ and $\sum_i \bar{h}_i = \bar{N}_c$. By following the same procedure as in the electric case (2.96), we can extract again the corresponding magnetic expectation values

$$\mathrm{Tr}\, Y^j = \sum_{i=1}^{n} \bar{h}_i \bar{a}_i^j + \sum_{k=1}^{n-1} \widetilde{R}(e_k) \sum_{i=1}^{n} \frac{\bar{a}_i^j}{(\bar{a}_i - e_k)\bar{V}''(\bar{a}_i)} + \delta^j_{n+1} \frac{\bar{S}(2\bar{N}_c - n + 1)}{\bar{t}_n} + \cdots\,, \tag{2.107}$$

for $j = 1, \ldots, n+1$. By $\cdots$ we denote the higher terms in the semiclassical expansion.

Now we can check that our singlets $P(z)$ in (2.9) match the electric mesons (2.93)

$$\widetilde{Q} X^{j-1} Q = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{4m(a_i)V''(a_i)} + \cdots\,, \qquad P_l = -\sum_{i=1}^{n} \frac{\bar{a}_i^{l-1} \widetilde{m}(\bar{a}_i)\mu^2}{\bar{V}''(\bar{a}_i)}\,. \tag{2.108}$$

At the leading approximation in the electric semiclassical expansion (2.40), the match is ensured by the condition (2.104) that solves the singlet equations of motion, provided that the relation between electric and magnetic polynomials and the quantum deformations is again

$$\overline{V}'(z) = -V'(z)\,, \qquad \bar{m}(z) = m(z), \qquad \overline{f}(z) = f(z)\,, \tag{2.109}$$

just the same we found in the massive case (2.83). Therefore electric and magnetic theory still have the same curve (2.84). Note that this is equivalent to the simple relation between the glueballs

$$\bar{S}_i = -S_i\,.$$

At this point we can rewrite the solution (2.104) of the magnetic theory is terms of the electric quantities, recalling that $\widetilde{m}(z)$ in (2.46) reverses its sign

$$4\mu^2 m(a_i)\widetilde{m}(a_i) = f(a_i)\,. \tag{2.110}$$

JHEP11(2004)020

The Konishi anomaly. Consider the low energy theory described by the vacuum (2.4). It is a product of decoupled SQCDs with N_f flavors. We can look at the physics of each separate $\mathrm{U}(N_i)$ SQCD by integrating the resolvents around the contour A_i, that encircles the branch points of the resolvent appeared by the splitting of the a_i root. In particular, by (2.93) the mesons, even if classically vanishing, at the quantum level satisfy the Konishi anomaly

$$\langle \widetilde{Q}Q\rangle_i = \frac{S_i}{m(a_i)} = -\frac{f(a_i)}{4m(a_i)V''(a_i)}\,, \tag{2.111}$$

where we dropped the higher terms in the semiclassical expansion (2.40).

We can perform a similar analysis in the magnetic theory: the electric mesons in (2.111) correspond to the gauge singlets P_1. The general expression for the singlets is given in (2.47). The low energy magnetic theory is a product of decoupled n Seiberg blocks, each one dual to a corresponding electric SQCD. The relation corresponding to the Konishi anomaly (2.111) in each low energy block is

$$\langle P_1\rangle_i = -\mu^2 \frac{\widetilde{m}(\bar{a}_i)}{\bar{V}\,''(\bar{a}_i)}\,, \tag{2.112}$$

and it matches the electric one due to the relations (2.104) and (2.109).

3. The electric Higgs phase

In this section we will find the classical magnetic solution when the electric theory is in the Higgs vacuum and check its properties. Then we will solve the chiral rings and look for the duality map. In the end we will consider the analytic structure of the solution we found as well as its behaviour when moving poles between the sheets. Our notations will be as follows: in sections 3.1 we will describe the classical magnetic theory with electric couplings, while in section 3.2 we will distinguish explicitly electric and magnetic couplings.

3.1 The classical vacua

The electric theory. To be definite we will consider the case in which only one flavor, e.g. the last one, is higgsed. We will begin by considering the classical theory with the simple KSS perturbation, namely the special case of (2.1) with

$$W_{el} = \frac{t_n}{n+1}\,\mathrm{Tr}\,X^{n+1} + m_2\widetilde{Q}_{N_f}XQ^{N_f} + m_1\widetilde{Q}_{N_f}Q^{N_f}\,. \tag{3.1}$$

This theory does not confine in the IR. Instead, the superpotential (3.1) drives the flow to an interacting SCFT. The Higgs vacuum in the electric theory is obtained by giving a classical expectation value to the last flavor of fundamentals

$$\widetilde{Q}_{N_f\alpha} = (\widetilde{h},0,\ldots,0)\,, \qquad Q^{N_f}_{\alpha} = (h,0,\ldots,0)\,, \tag{3.2}$$

then the quark equations of motion fix the value of the adjoint to $X = \mathrm{diag}(x_1,0,\ldots,0)$ where $x_1 = -m_1/m_2$. The expectation value of the quarks is fixed by the adjoint equations of motion to $h\widetilde{h} = -t_n x_1^n/m_2$.

As usual, we can think of the low energy theory in two stages. First, by higgsing the quarks we decrease the number of colors from N_c to N_c-1. The quarks $Q^{\alpha}_{N_f}$, $\widetilde{Q}^{N_f}_{\alpha}$ for $\alpha=2,\dots,N_c$ become the transverse component of a massive vector superfield of mass squared $\widetilde{h}h$, while the components X^1_{α}, X^{α}_1 for $\alpha=2,\dots,N_c$ of the adjoint replace the last flavor, so the total number of flavors does not decrease. Secondly, this latter new flavor acquires a mass $t_n x_1^{n-1}$ by expanding the adjoint superpotential. The low energy theory is a $\mathrm{U}(N_c-1)$ gauge theory with N_f-1 flavors. The matching of the electric scales is

$$\Lambda^{2N_c-N_f}_{N_c,N_f}=\frac{(-x_1)}{m_2}\Lambda^{2(N_c-1)-(N_f-1)}_{N_c-1,N_f-1}\,. \tag{3.3}$$

The magnetic theory. The magnetic theory corresponding to (3.1) is defined by

$$W_{\mathrm{mag}}=-\frac{t_n}{n+1}\,\mathrm{Tr}\,Y^{n+1}+\frac{t_n}{\mu^2}\sum_{j=1}^{n}P_j\widetilde{q}Y^{n-j}q+m_2(P_2)^{N_f}_{N_f}+m_1(P_1)^{N_f}_{N_f}\,. \tag{3.4}$$

Unlike the previous pseudoconfining case, this vacuum is characterized by a nonvanishing classical expectation value for the singlets, corresponding to the electric higgsed quarks, that we classically match as

$$(P_j)^{N_f}_{N_f}=\widetilde{Q}_{N_f}X^{j-1}Q^{N_f}=-\frac{t_n x_1^n}{m_2}x_1^{j-1}\,, \tag{3.5}$$

for $j=1\dots,n$. In this case the classical chiral ring is more complicated, due to the nonvanishing singlets. In addition to the usual singlet equations of motion (2.10), we have also the quark

$$\sum_{j=1}^{n}P_j(Y^{n-j}q)=0,\quad \sum_{j=1}^{n}P_j(\widetilde{q}Y^{n-j})=0\,, \tag{3.6}$$

as well as the adjoint equations of motion. Nevertheless one can check that, in the convenient notation of (2.14) and (2.15), the adjoint is in block diagonal form $Y=\mathrm{diag}(Y_{\mathrm{Higgs}},0,\dots,0)$ and the nonvanishing part of the solution is

$$\begin{aligned}
Y_{\mathrm{Higgs}}&=|1\rangle\langle v|+b_2R_{n-1}\,,\\
|v\rangle&=-\sum_{j=1}^{n-1}x_1\left(\frac{x_1}{b_2}\right)^{j-1}|j\rangle\,,\\
|\widetilde{q}^{N_f}\rangle&=b_2|1\rangle,\qquad |q_{N_f}\rangle=b_2|n-1\rangle\,,
\end{aligned} \tag{3.7}$$

where b_2 is given by (2.12). The first $(n-1)\times(n-1)$ block of the adjoint reads

$$Y_{\mathrm{Higgs}}=\begin{pmatrix}
-x_1 & b_2 & 0 & & \\
-x_1(\frac{x_1}{b_2}) & 0 & b_2 & . & \\
. & & 0 & . & . \\
. & & & . & b_2 \\
-x_1(\frac{x_1}{b_2})^{n-2} & 0 & . & . & 0
\end{pmatrix} \tag{3.8}$$

Let us figure the low energy theory. First, by higgsing the theory we break the gauge symmetry as $\mathrm{U}(\bar{N}_c)\to\mathrm{U}(\bar{N}_c-n+1)$ and note that $\bar{N}_c-n+1=n(N_f-1)-(N_c-1)$, as expected from the electric theory. Accordingly, $\widetilde{q}^{N_f}_{\alpha}$, $q^{\alpha}_{N_f}$, Y^{α}_m, Y^s_{α} for $\alpha=n,\dots,\bar{N}_c$,

$m = 2, \ldots, n-1$ and $s = 1, \ldots, n-2$ conspire to join $n-1$ massive vector superfields in the fundamental of $\mathrm{U}(\bar{N}_c - n + 1)$ with mass b_2. The flavor that disappears is replaced by a new flavor Y_1^α, Y_α^{n-1} for $\alpha = n, \ldots, \bar{N}_c$, so the number of flavors does not decrease. Secondly, we look for a mass term for the new flavor coming from the superpotential

$$\frac{t_n}{n+1}\,\mathrm{Tr}\,Y^{n+1} \simeq t_n(-x_1) b_2^{n-2} Y_1^\alpha Y_\alpha^1\,, \qquad \alpha = n, \ldots, \bar{N}_c\,. \tag{3.9}$$

The number of flavors thus decreases by one unit also in the magnetic theory. The matching of the magnetic scale goes as follows

$$\widetilde{\Lambda}^{2\bar{N}_c - N_f}_{\bar{N}_c, N_f} = \frac{b_2^n}{t_n(-x_1)} \widetilde{\Lambda}^{2(\bar{N}_c - n+1)-(N_f-1)}_{\bar{N}_c - n+1, N_f - 1}\,. \tag{3.10}$$

We can use the relation (1.6) between the scales and find that this solution is consistent with the flows

$$\Lambda^{2(N_c-1)-(N_f-1)}_{N_c-1, N_f-1} \widetilde{\Lambda}^{2(\bar{N}_c - n+1)-(N_f-1)}_{\bar{N}_c - n+1, N_f-1} = \left(\frac{\mu^2}{t_n^2}\right)^{N_f - 1}\,. \tag{3.11}$$

Generic polynomial deformation. We can generalize this to an arbitrary polynomial deformation

$$W_{el} = \mathrm{Tr}\,V(X) + m_2 \widetilde{Q}_{N_f} X Q^{N_f} + m_1 \widetilde{Q}_{N_f} Q^{N_f}\,. \tag{3.12}$$

The classical solution is the same as in (3.2) the only difference being that now $\widetilde{h}h = -V'(x_1)/m_2$.

In the magnetic theory, the corresponding solution is as in (3.7) but now the vector $|v\rangle$ is replaced by

$$|v'\rangle = -\sum_{j=1}^{n-1} \left(\frac{x_1}{b_2}\right)^{j-1} \left(x_1 + \sum_{l=0}^{j-1} x_1^{-l} \frac{t_{n-l-1}}{t_n}\right) |j\rangle\,. \tag{3.13}$$

Now that we have the generic adjoint polynomial $V(z)$, we can keep on flowing by further breaking the gauge group down to the low energy SQCD blocks. The electric adjoint is then $X = \mathrm{diag}(x_1, a_1, \ldots, a_n)$, where a_i are the roots of $V'(z)$ that appear with multiplicity N_i such that $\sum_i N_i = N_c - 1$. In the magnetic theory we have correspondingly a bunch of diagonal blocks $Y = \mathrm{diag}(Y_{\mathrm{Higgs}}, Y_{a_1}, \ldots, Y_{a_n})$, where the first one is (3.13) and the others are as in (2.17). In this vacuum the relation between the low energy electric and magnetic gauge groups is

$$\bar{N}_i = N_f - N_c - 1\,, \tag{3.14}$$

since in the higgsed electric theory we have $\sum_{i=1}^n N_i = N_c - 1$. We can also compute the classical expression of the generators of the chiral ring operators in this vacuum. The resolvent $\widetilde{R}(z)$ vanishes, while

$$\begin{aligned} \widetilde{M}_{\mathrm{cl}}(z) &= -\mu^2 \frac{m(z)}{V'(z) - V'(x_1)}\,, \\ \widetilde{T}_{\mathrm{cl}}(z) &= \sum_{i=1}^n \frac{\bar{N}_i}{z - a_1} + \frac{d}{dz} \ln \frac{V'(z) - V'(x_1)}{z - x_1}\,. \end{aligned} \tag{3.15}$$

Note that (3.15) gives the correct behaviour at infinity $T_{\mathrm{cl}} \sim \bar{N}_c/z$ since $\sum_{i=1}^n \bar{N}_i = \bar{N}_c - n + 1$.

Let us mention that this description is agreement with the expectations from electric magnetic duality. If we compare this solution to the pseudoconfining one (2.15), we see that while the electric theory, being higgsed, becomes more weakly coupled, in the magnetic theory the rank of the higgsed block in the adjoint decreases from n to $n-1$, thus making the theory more strongly coupled.

One can find a small generalization of the solution (3.7) by turning on higher meson perturbations in the electric theory, always along the last electric flavor direction. In appendix C we will give more details about the solutions with several higgsed electric colors, but now let us add just few comments. In this way we can have more higgsed entries in the same flavor $Q^{N_f} = (h_1, \ldots, h_l, 0, \ldots, 0)$ and correspondingly $X = \mathrm{diag}(x_1, \ldots, x_l, 0, \ldots, 0)$, the electric theory being at weaker coupling. The general structure of the magnetic expectation values is that the first Y_{Higgs} block decreases its rank down to $n-l$. Hence, the magnetic side looks more strongly coupled. The rank of the generic Seiberg blocks is still $\bar{N}_i = N_f - N_i - 1$ and one can check that still

$$\sum_{i=1}^{n}(N_f - N_i - 1) + n - l = nN_f - N_c = \bar{N}_c\,, \tag{3.16}$$

since now $\sum_{i=1}^{n} N_i = N_c - l$. We can carry on this procedure until $l = n-1$: one further higgsing would get the rank of Y_{Higgs} vanished. In fact $l = n-1$ is also the maximal number of Higgs eigenvalues we can turn on on the same electric flavor, i.e. the largest value the degree of the meson polynomial $m(z)$ can reach, higher mesons being trivial in the electric chiral ring. Finally, if we allow for different electric flavors to get higgsed then in the magnetic theory we have to add a new block analogous to Y_{Higgs} for each higgsed flavor. We can not go on higgsing forever, issues similar to the one that led our discussion of (2.29) arise also in this phase.

3.2 The chiral ring

The electric theory. Let us consider the minimal case in which the electric theory admits a Higgs vacuum and we can safely apply the DV method: all flavors are massive and a Yukawa interaction is turned on only for the last flavor. The tree level superpotential is

$$W_{\mathrm{tree}} = \mathrm{Tr}\, V(X) + m_1 \widetilde{Q}_f Q^f + m_2 \widetilde{Q}_{N_f} X Q^{N_f}\,, \tag{3.17}$$

so that the meson polynomial reads $m(z)^{\tilde{f}}_f = m_1 \delta^{\tilde{f}}_f + z\, m_2 \delta^{N_f}_f \delta^{\tilde{f}}_{N_f}$ and has only one root $x_1 = -m_1/m_2$. We give a classical expectation value to the last flavor of quarks and consider the following solution to the equations of motion

$$\begin{aligned} X &= \mathrm{diag}(x_1, a_1, \ldots, a_n) \\ \widetilde{Q}_{N_f} &= (\tilde{h}_1, 0, \ldots, 0)\,, \qquad Q^{N_f} = (h_1, 0, \ldots, 0)\,, \end{aligned} \tag{3.18}$$

where each a_i is a root of $V'(z)$ and has multiplicity N_i such that $\sum_{i=1}^{n} N_i = N_c - 1$. The adjoint equations of motion set $\tilde{h}_1 h_1 = -V'(x_1)/m_2$. In the classical chiral ring the resolvent $R(z)$ vanishes, while the nonvanishing generators are

$$\begin{aligned} T(z)|_{\rm cl} &= \frac{1}{z-x_1} + \sum_{i=1}^{n} \frac{N_i}{z-a_i}\,, \\ M_{N_f}^{N_f}(z)|_{\rm cl} &= -\frac{V'(x_1)}{z-x_1}\oint_{x_1}\frac{dx}{m_{N_f}^{N_f}(x)} = -\frac{1}{m_2}\frac{V'(x_1)}{z-x_1}\,, \end{aligned} \tag{3.19}$$

Let us solve the anomaly equations. The resolvent $R(z)$ is always (2.32). The story is different now for the generator of the mesons $M(z)$. The boundary conditions in the Higgs vacuum require a pole on the first sheet along the last flavor direction. The solution along the pseudoconfining flavor directions is the usual one

$$M(z)_f^{\tilde f} = R(z) m_1^{-1}\delta_f^{\tilde f}, \qquad (f,\tilde f) \neq (N_f, N_f)\,, \tag{3.20}$$

while the solution along the last flavor direction is

$$M(z)_{N_f}^{N_f} = \frac{R(z)}{m_1 + z\, m_2} - \frac{V'(x_1) - R(x_1)}{z - x_1} m_2^{-1}\,. \tag{3.21}$$

We can integrate (3.21) on the contour A that encircles all the branch points of the resolvent and obtain the quantum expressions for the mesons. There are two types of mesons, the ones in the $(f,\tilde f) \neq (N_f,N_f)$ flavor directions that are exactly given by (2.39), and the ones in the last flavor direction that are

$$\widetilde{Q}_{N_f} X^{j-1} Q^{N_f} = -\sum_{i=1}^{n} \frac{a_i^{j-1} f(a_i)}{(m_1 + a_i m_2) V''(a_i)} + \cdots\,, \tag{3.22}$$

where the dots stand for higher terms in the semiclassical expansion of the resolvent (2.40).

The magnetic theory and its analytic structure. The magnetic theory corresponding to (3.17) is defined by the following tree level superpotential

$$W_{\rm mag} = \operatorname{Tr}\bar V(Y) + \tilde q^{\tilde f}\bar m(Y)_{\tilde f}^{f} q_f + \bar m_1 \operatorname{tr} P_1 + \bar m_2 (P_2)_{N_f}^{N_f}\,, \tag{3.23}$$

The anomaly equation for the resolvent $\widetilde R(z)$ gives the usual solution (2.48). The equations for $\widetilde M(z)$ and the singlet equations of motion now have different boundary conditions depending on the flavor directions. The first $(f,\tilde f) \neq (N_f,N_f)$ flavors have the same solution (2.53) and (2.54) as in the first massive case we considered, in which $\widetilde m(z)$ is proportional to the quantum deformation

$$\begin{aligned} \widetilde M(z)_f^{\tilde f} &= \widetilde R(z)\widetilde m(z)^{-1}{}_f^{\tilde f}\,, \\ \widetilde m(z)_{\tilde f}^{f} &= -\frac{\bar f(z)}{4\mu^2}(\bar m^{-1})_{\tilde f}^{f}. \end{aligned} \tag{3.24}$$

The remaining flavor direction $(f,\tilde f) = (N_f,N_f)$ corresponds to the higgsed electric meson. The new boundary conditions for $\widetilde M(z)$ are $n-1$ poles on the first sheet and no pole on the second sheet, as opposed to the previous pseudoconfining case (2.98) in which no pole was there on the first sheet and $n-1$ poles appeared on the second sheet

$$\widetilde{M}(z)^{N_f}_{N_f} = \frac{\widetilde{R}(z)}{\widetilde{m}(z)^{N_f}_{N_f}} - \sum_{i=1}^{n-1} \frac{\bar{V}'(e_k) - \widetilde{R}(e_k)}{z - e_k} \frac{1}{\widetilde{m}'(e_k)^{N_f}_{N_f}}, \tag{3.25}$$

where e_k for $k = 1, \ldots, n-1$ are the roots of $\widetilde{m}(z)$.

The picture of the analytic structure of $\widetilde{M}(z)$ is the following. We saw that in the pseudoconfining electric case (2.4), the magnetic solution (2.98) does not have poles on the first sheet, but it has $n-1$ poles on the second sheet. In the classical limit, these poles are very large, but in the quantum theory they move to the region near the branch cuts, as we will check explicitly in section 4. In the electric Higgs phase (3.2), the magnetic solution (3.25) gets $n-1$ poles on the first sheet and no pole on the second sheet. The classical limit of this last solution has still $n-1$ poles, coming from the second term in (3.25) and the fact that classically $\widetilde{m}$ is nonvanishing. Now let us move back to the electric theory and Higgs two color direction on the same electric flavor, replacing (3.18) with

$$\widetilde{Q}_{N_f} = (\widetilde{h}_1, \widetilde{h}_2, 0, \ldots, 0), \qquad Q^{N_f} = (h_1, h_2, 0, \ldots, 0). \tag{3.26}$$

and $X = \mathrm{diag}(x_1, x_2, 0, \ldots, 0)$. The gauge group is higgsed down to $\mathrm{U}(N_c - 2)$ and the electric theory becomes more weakly coupled. Classically we saw in (3.16) that the rank of the corresponding magnetic Higgs block decreases by one. Quantum mechanically this corresponds to moving one of the $n-1$ poles in (3.25) from the first to the second sheet

$$M(z)^{N_f}_{N_f} = \frac{R(z)}{\widetilde{m}(z)} - \sum_{i=2}^{n-1} \frac{V'(e_k) - R(e_k)}{z - e_k} \frac{1}{\widetilde{m}'(e_k)} - \frac{R(e_1)}{(z - e_1)} \frac{1}{\widetilde{m}'(e_1)}. \tag{3.27}$$

In this way the magnetic theory becomes more strongly coupled. In the classical limit $\widetilde{m}(z)$ is nonvanishing so we are left with just $n-2$ poles. Note that we can higgs at most $n-1$ electric color directions on the same flavor, $Q^{N_f} = (h_1, \ldots, h_{n-1}, 0, \ldots, 0)$, corresponding to the largest degree the electric meson polynomial $m(z)$ can have. On the magnetic side, there are at most $n-1$ poles to be moved all the way to the second sheet. When we pass them all, the corresponding meson generator looks much like (2.98), but actually it is different. While the classical limit of (2.98) is nonzero due to the fact that $\widetilde{m}(z)$ vanishes classically, in this case $\widetilde{m}(z)$ is always nonvanishing and therefore the meson generator vanishes classically.

In the previous pseudoconfining case, we noted that, even if classically $\widetilde{M}(z)$ has some singularities, in the quantum theory these singularities are smoothed out and we end up with a regular expression in the first semiclassical sheet. In the Higgs case, instead, the singularities we might expect in the classical generator do not disappear at the quantum level but are genuine poles in the quantum expressions (3.25).

We still have to fix $\widetilde{m}(z)$ by requiring that the singlet equations of motion (2.103) are satisfied. The contour $\widetilde{A}$ in (2.103) encircles all the branch points of the resolvent, but now it encircles also the $n-1$ poles at e_k. The evaluation of this contour integral is much more complicated than in the pseudoconfining case (2.103), since we get additional residues at e_k. Dropping higher terms in the semiclassical expansion of the resolvent (2.40) and showing just the leading approximation we get

$$\sum_{i=1}^{n}\left[-\frac{\bar{a}_i^{l-1}\bar{f}(\bar{a}_i)}{4\widetilde{m}(\bar{a}_i)\bar{V}''(\bar{a}_i)}-\mu^2\frac{\bar{a}_i^{l-1}\bar{m}(\bar{a}_i)}{\bar{V}''(\bar{a}_i)}\right]+\sum_{k=1}^{n-1}\frac{2\widetilde{R}(e_k)-\bar{V}'(e_k)}{\widetilde{m}'(e_k)}e_k^{l-1}=0\,, \tag{3.28}$$

for $l=1,\dots,n$. Again we see that (3.28) amounts to n conditions that implicitely fix the unknown polynomial $\widetilde{m}(z)$. However, in this case it is hard to solve these equations explicitly since the roots e_k appear inside the resolvent.

Now consider the matching (2.81) between the gauge singlets and the electric mesons. The mesons in the directions $(f,\widetilde{f})\neq(N_f,N_f)$ match as in the first massive case, reobtaining the map

$$\bar{V}'(z)=-V'(z)\,,\qquad \bar{f}(z)=f(z)\,,\qquad m_f^{\widetilde{f}}=\bar{m}_f^{\widetilde{f}}\,. \tag{3.29}$$

The last direction $(f,\widetilde{f})=(N_f,N_f)$ gives a new condition, that we can write as

$$\oint_{A'} z^{l-1}\left[M_{el}(z)_{N_f}^{N_f}+\mu^2\frac{\widetilde{m}(z)_{N_F}^{N_f}}{V'(z)}\right]=0\,, \tag{3.30}$$

for $l=1,\dots,n$. The electric meson generator is given by (3.21) and we replaced the magnetic adjoint polynomial with the electric one by (3.29). The contour A' now is a very large contour that encircles the branch points of the resolvent as well as the electric pole at the point x_1 in the first sheet. Evaluating the contour integral at first order in the semiclassical expansion and dropping the higher terms we find

$$\sum_{i=1}^{n}\frac{a_i^{l-1}f(a_i)}{4m(a_i)V''(a_i)}-\frac{2R(x_1)-V'(x_1)}{m_2}x_1^{l-1}+\sum_{i=1}^{n}\mu^2\frac{a_i^{l-1}\widetilde{m}(a_i)}{V''(a_i)}=0\,, \tag{3.31}$$

for $l=1,\dots,n$.

Had we not allowed the contour to encircle the pole at x_1, this expression would have had an inconsistent classical limit. Let us consider in fact the classical limit of the conditions we have found so far. This is achieved by setting to zero the quantum deformation $f(z)$ so that the resolvent vanishes in the first sheet. It is more transparent to write the two classical conditions as contour integrals. We fix the classical polynomial $\widetilde{m}_{\mathrm{cl}}(z)$ by the singlet equations

$$\oint_{A_{\mathrm{cl}}} z^{l-1}\left[\mu^2\frac{m(z)}{V'(z)}+\frac{V'(z)}{\widetilde{m}(z)}\right]=0\,, \tag{3.32}$$

for $l=1,\dots,n$, where the contour encircles all the poles of the two meromorphic functions. By picking up the residues we get

$$\mu^2\sum_{i=1}^{n}\frac{a_i^{l-1}m(a_i)}{V''(a_i)}+\sum_{k=1}^{n}\widehat{e}_k^{\,l-1}\frac{V'(\widehat{e}_k)}{\widetilde{m}'_{\mathrm{cl}}(\widehat{e}_k)}=0\,, \tag{3.33}$$

where we hatted the classical roots $\widehat{e}_k$. This condition is much easier to solve than (3.28) due to the disappearance of the resolvent. Once we fix $\widetilde{m}_{\rm cl}(z)$, the classical limit of the matching condition (3.30) is satisfied

$$\oint_{A'_{\rm cl}} z^{l-1}\left[\mu^2\frac{\widetilde{m}(z)}{V'(z)}+\frac{V'(z)}{m(z)}\right]=0\,, \tag{3.34}$$

for $l=1,\dots,n$, whose evaluation yields

$$\mu^2\sum_{i=1}^{n}\frac{a_i^{l-1}\widetilde{m}_{\rm cl}(a_i)}{V''(a_i)}+x_1^{l-1}V'(x_1)m_2^{-1}=0\,. \tag{3.35}$$

4. The cubic superpotential

In this section we will illustrate the pseudoconfining and Higgs phase computations, worked out in the previous sections, in the simplest example that allows for a Higgs phase, namely a cubic interaction for the adjoint.

4.1 The pseudoconfining case

Let us consider an electric tree level superpotential as in (3.17) and let us specialize to $n=2$. We take the following adjoint polynomial

$$V'(z)=t_1z+t_2z^2\,,$$

whose roots are $a_1=0$ and $a_2=-t_1/t_2$. We also have a meson polynomial $m(z)_f^{\tilde{f}}=m_1\delta_f^{\tilde{f}}+z\,m_2\delta_f^{N_f}\delta_{N_f}^{\tilde{f}}$. The resolvent is $2R(z)=V'(z)-\sqrt{V'(z)^2+f(z)}$ and its quantum deformation is $f(z)=f_0+f_1z$.

In the magnetic theory, all the flavor directions $(f,\tilde{f})\neq(N_f,N_f)$ correspond to the massive case solved in section 2.4. In the following we will focus instead on the last direction $(f,\tilde{f})=(N_f,N_f)$ only and suppress the flavor indices. We will see an explicit example of the computations in sections 2 and 3. Let us consider $\widetilde{m}(z)=\widetilde{m}_1+\widetilde{m}_2z$, whose one root we denote as $e_1=-\widetilde{m}_1/\widetilde{m}_2$. The solution (2.110) of the magnetic theory is given by the condition $4\mu^2m(a_i)\widetilde{m}(a_i)=f(a_i)$ for $i=1,2$, from which we get the $\widetilde{m}(z)$ coefficients in terms of $m(z)$ and $f(z)$

$$\begin{aligned}\widetilde{m}_1&=\frac{f_0}{4\mu^2m_1}\,,\\ \widetilde{m}_2&=-\frac{t_2}{4\mu^2t_1}\left(\frac{t_2f_0-t_1f_1}{t_2m_1-t_1m_2}-\frac{f_0}{m_1}\right),\end{aligned} \tag{4.1}$$

so that the singlet equations of motion (2.10) are satisfied. This condition also ensures that the singlets P_j extracted from (2.9) match the electric mesons $\widetilde{Q}X^{j-1}Q$.

We would like to check that the root e_1 lies inside the contour $\widetilde{A}$ that encircles the branch points of the resolvent. Consider the classical limit of this setup. In this limit both $f(z)$ and $\widetilde{m}(z)$ vanish, but we still have to satisfy the singlet equations of motion. We first want to obtain the dependence of $f(z)$ on the total glueball and then perform the limit by sending the glueball to zero. For this purpose we have to choose a vacuum for the electric theory and solve the factorization of gauge theory curve. Let us consider the phase in which the gauge group is unbroken, which corresponds to the one-cut case, namely the electric adjoint is $X = \mathrm{diag}(a_i, \ldots, a_i)$. Then the curve factorizes as

$$V'(z)^2 + f(z) = t_2^2(z-k)^2(z-a+b)(z-a-b)\,, \tag{4.2}$$

with one double root and two branch points. We already know from (2.37) that $f_1 = -4t_2 S$ and we can find [17]

$$\begin{aligned} k &= -\frac{t_1}{t_2} + a\,, \\ a &= \frac{t_2}{t_1}S + \mathcal{O}(S^2)\,, \\ b &= \sqrt{\frac{S}{2m}}\,(2 + \mathcal{O}(S))\,. \end{aligned} \tag{4.3}$$

We don't need the full result, but just the leading terms in the glueball, from which we find $f_0 = -2t_1 S + \mathcal{O}(S^2)$. Then in the classical limit $S \to 0$ we have

$$\frac{f_0}{f_1} \sim \frac{t_1}{t_2} + \mathcal{O}(S)\,. \tag{4.4}$$

The root e_1 of $\widetilde{m}(z)$ in the classical limit is

$$\widehat{e}_1 = \frac{t_1}{t_2}\frac{t_2 m_1 - t_1 m_2}{t_1 m_2 - 2t_2 m_1}\,. \tag{4.5}$$

In the limit of large mass m_1 for the electric quarks, we find $\widehat{e}_1 \sim -\frac{t_1}{2t_2}$, which is not large but lie inside the contour $\widetilde{A}$ that encircles the branch points of the resolvent, as we claimed below eq. (2.103). In particular, this classical pole is halfway between the two roots of $V'(z)$.

4.2 The Higgs case

We keep the same superpotential, but consider now the electric Higgs vacuum $X = \mathrm{diag}(x_1, 0, \ldots, 0)$ and

$$\widetilde{Q}_{N_f} = (\widetilde{h}_1, 0, \ldots, 0), \qquad Q^{N_f} = (h_1, 0, \ldots, 0)\,, \tag{4.6}$$

where the gauge group is higgsed down to $\mathrm{U}(N_c - 1)$ and the electric equations of motion set $\widetilde{h}_1 h_1 = -V'(x_1)/m_2$. This vacuum is characterized by a nonvanishing classical expectation value for the electric mesons

$$\widetilde{Q}_{N_f} X^{j-1} Q^{N_f} = (P_j)^{N_f}_{N_f} = -\frac{x_1^{j-1} V'(x_1)}{m_2}\,. \tag{4.7}$$

We want to check the prescription we outlined in section 3.2 in the magnetic theory. In the Higgs phase the singlets $P(z)$ as well as the magnetic polynomial $\widetilde{m}(z)$ acquire a classical expectation value. From (4.7) we can read out their classical expressions

$$\begin{aligned} P(z)_{\rm cl} &= \frac{\bar{V}'(\bar{x}_1)}{\bar{m}_2 z^2}(z+\bar{x}_1)\,, \\ \widetilde{m}(z)_{\rm cl} &= \frac{\bar{V}'(\bar{x}_1)}{\mu^2 \bar{m}_2}(t_1+t_2\bar{x}_1+zt_2)\,. \end{aligned} \tag{4.8}$$

Now we would like to solve the quantum theory at first order in the semiclassical expansion. If we look at the flavor directions $(f,\widetilde{f}) \neq (N_f,N_f)$ we find the duality map (3.29), that we can use in the following computation. In the higgsed direction, first we have to solve the singlet equations of motion (3.28) and then check that the matching relation (3.31) is satisfied. But this is kind of hard, due to the presence of the resolvent in the last term of (3.28) that makes the equations pretty much involved. However, since the solutions of (3.31) must be solutions of (3.28) too, the best we can do is we solve the matching condition (3.31) and then try to check that this solution satisfies the singlet equations of motion (3.28), thus getting it the other way around.

The matching condition at first order is

$$\sum_{i=1}^{2}\left[-\frac{a_i^{l-1}f(a_i)}{4\bar{m}(a_i)V''(a_i)}-\mu^2\frac{a_i^{l-1}\widetilde{m}(a_i)}{V''(a_i)}\right]+\frac{2R(x_1)-V'(x_1)}{\bar{m}_2}x_1^{l-1}=0\,, \tag{4.9}$$

for $l=1,2$. This can be solved easily with the result

$$\begin{aligned} \widetilde{m}_1 &= -\frac{f_0-4V'(\bar{x}_1)[V'(\bar{x}_1)-2R(\bar{x}_1)]}{4\mu^2\bar{m}_1}\,, \\ \widetilde{m}_2 &= \frac{t_2}{4\mu^2\bar{m}_2V'(\bar{x}_1)}\left[f(\bar{x}_1)-4V'(\bar{x}_1)[V'(\bar{x}_1)-2R(\bar{x}_1)]\right]. \end{aligned} \tag{4.10}$$

Quantum mechanically, the one root of $\widetilde{m}(z)$ is $e_1=-\widetilde{m}_1/\widetilde{m}_2$. If we take the classical limit of (4.10) we obtain the expected expression (4.8) and its classical root $\widehat{e}_1=-x_1-t_1/t_2$, by identifying $\bar{m}(z)=m(z)$. In the semiclassical electric picture in which the Higgs vev x_1 is large, this root gets very large, too. This phase is very different from (4.5), where for large electric quark masses we got small $\widehat{e}_1$.

It would be very hard to check that (4.10) satisfies the singlet equations of motion (3.28) at first order in the semiclassical expansion, due to the fact that the resolvent should be evaluated at the root of $\widetilde{m}(z)$. But one can still easily check that indeed the classical limit of the singlet equations

$$\mu^2\sum_{i=1}^{2}\frac{a_i^{l-1}m(a_i)}{V''(a_i)}+\widehat{e}_1^{\,l-1}\frac{V'(\widehat{e}_1)}{\widetilde{m}'_{\rm cl}(\widehat{e}_1)}=0\,, \qquad l=1,2\,, \tag{4.11}$$

is satisfied by $\widehat{e}_1$ and the classical limit of (4.10).

JHEP11(2004)020

5. Discussion

Let us summarize our results and suggest some further speculations. At the classical level, we generalized the KSS solution to the case of polynomial superpotentials, allowing for generic meson deformations, and we found the solutions of the magnetic theory corresponding to the electric pseudoconfining and Higgs vacua. We considered then duality in the quantum theory and we used the DV approach to solve for the chiral rings just above the mass gap: we studied the effective glueball superpotential. We analyzed the following three cases:

1. The electric meson superpotential is a mass term for all the flavors. We saw that electric-magnetic duality holds exactly offshell in this case.

2. The generic pseudoconfining phase, where we allow for a generic meson deformation, has a way richer analytic structure. We matched the electric mesons with the magnetic singlets at first order in the semiclassical expansion of the resolvent. In this way we found a condition that reproduces the Konishi anomaly equation in the low energy SQCD blocks and their magnetic dual. In this case duality does not hold exactly offshell.

3. In the electric Higgs phase, we found the solution to the magnetic theory, at first order in the semiclassical expansion, and showed that it is consistent with the classical limit. Neither in this case does duality work exactly offshell. Moreover, while in the pseudoconfining case the classical singularities in $\widetilde{M}(z)$ are just an artifact of the classical solution and in the quantum theory they disappear, in the Higgs case the classical singularities are preserved in the quantum theory.

As we just summarized, electric-magnetic duality holds exactly offshell in some cases, namely in SQCD or when the meson polynomial $m(z)$ is z-independent, while in more general cases it should work exactly only onshell. It would be nice to find a physical motivation for such different behaviors.

We could draw a picture of the analytic properties of the magnetic theory as we continuously interpolate between different Higgs vacua in the electric theory (when we move poles from the second to the first electric sheet). An interesting extension of our analysis would be to show what happens on the magnetic side when we smoothly pass from the pseudoconfining to the Higgs phase in the electric theory. In this way, one might shed some light on the onshell process that takes place when a branch cut of the resolvent closes up, as recently investigated in [18]. On the electric side this is a strong coupling phenomenon, but one should describe it easily in the dual regime.

On the other hand, it would be interesting to use our quantum duality map to gain insight on the meaning of the electric parameter L introduced in [11] as the degree of the determinant of the meson polynomial $B(z) = \det m(z)$. This parameter plays the role of an effective number of flavors and is related to the appearence of instanton corrections to the classical chiral ring. In particular, if the electric superpotential $V(z)$ has degree $N_c + 1$, when $L \geq N_c$ the strong coupling analysis shows that the classical Casimirs $\mathrm{Tr}\, X^j$

for $j = 1, \ldots, N_c$ are modified in the quantum chiral ring by terms proportional to the instanton factor. It would be interesting to understand the corresponding phenomenon in the magnetic theory. In our setup, $L \leq N_f(n-1)$, so the condition for the appearence of instanton corrections is related to $N_f \leq \bar{N}_c$ on the magnetic side.

A natural generalization of our analysis would be to consider $\mathrm{SO}(N_c)$ and $\mathrm{Sp}(2N_c)$ gauge groups. In particular, one could translate into a magnetic language the map between $\mathrm{Sp}(2N_c)$ theory with an antisymmetric tensor and $\mathrm{U}(2N_c+2n)$ with an adjoint, recently proposed in [19]. Moreover, the KSS duality has been generalized to theories with two adjoint chiral superfields and fundamentals in [20]. It would be nice to extend our classical and quantum mechanical solutions to this case. One might find some unusual features, due to the fact that the gauge theory curve is not hyperelliptic anymore in this theory and cannot be obtained as a deformation of an $\mathcal{N}=2$ theory.

We would like to make one last remark on the theory without superpotential, whose magnetic dual is not known. In [9] it was suggested that one might try to obtain this theory as a certain limit of the KSS theory with superpotential $t_n \operatorname{Tr} X^{n+1}$. Since the limit of vanishing t_n is singular, it was suggested to study the $k \to \infty$ limit instead, so that the magnetic dual might look like an $\mathrm{U}(\infty)$ gauge theory, which is expected to behave like a string theory. The story might be simpler, though. Due to the recent work of Intriligator and Wecht [21], we know that an analogue of the conformal window of SQCD exists also for the KSS theory: it is the region in the range of N_f in which both the electric and the magnetic deformations $\operatorname{Tr} X^{n+1}$ and $\operatorname{Tr} Y^{n+1}$ are relevant [22]. Now, if we take a sufficiently large number of flavors we can make the deformation $\operatorname{Tr} X^{n+1}$ irrelevant, but still keeping the electric theory asymptotically free. Therefore, the electric theory at the fixed point will be the theory without superpotential. But on the magnetic side, the corresponding superpotential keeps being relevant and we have the usual full magnetic theory. So we might not really need to take k very large to remove the electric superpotential, hence the magnetic dual of the theory without superpotential need not be a kind of string theory. This point might deserve further study.

Acknowledgments

I would like to thank A. Schwimmer for help at every stage of this project. It is also a pleasure to thank L. Alday, M. Bertolini, L. Bonora, F. Cachazo, M. Cirafici, J. David, M. Matone and especially K. Intriligator and N. Seiberg for discussions and suggestions. I would like to thank the organizers of the PiTP 2004 at the Institute for Advanced Studies, Princeton, for their kind hospitality and for providing a very stimulating environment, where this work has been completed.

A. The DV method

In the following we will quote some results on the generalized Konishi anomaly approach to DV, following [4, 10, 11]. We will collect some useful formulae we need in the main part of the paper. For a review of the DV approach and an extensive discussion of the huge literature available by now see [15] and references therein.

JHEP11(2004)020

Consider the electric theory with tree level superpotential (2.1) and specialize to a diagonal meson polynomial $m(z)_{f}^{\widetilde{f}} = m(z)\delta_{f}^{\widetilde{f}}$ of degree $l \leq n-1$. Classically, we can consider the following generic vacuum

$$
\begin{aligned}
X &= \mathrm{diag}(x_1, \dots, x_i, a_1, \dots, a_n)\,, \\
\widetilde{Q}_{N_f} &= (\widetilde{h}_1, \dots, \widetilde{h}_i, 0, \dots, 0)\,, \\
Q^{N_f} &= (h_1, \dots, h_i, 0, \dots, 0)
\end{aligned} \tag{A.1}
$$

where $i \leq l$ and the adjoint equations of motion fix $\widetilde{h}_i h_i = -V'(x_i)/m'(x_i)$. We will Higgs only the last flavor of quarks for at most l color directions. We will introduce the occupation number r_I for each root x_k of the meson polynomial: $r_k = 1$ if the root appears in the adjoint in (A.1) and $r_k = 0$ otherwise. We will be interested in the chiral operators (2.3). Their classical expressions are

$$
\begin{aligned}
M(z)_{\mathrm{cl}} &= -\sum_{k=1}^{l} \frac{r_k V'(x_k)}{z - x_k} \frac{1}{m'(x_k)}\,, \\
T(z)_{\mathrm{cl}} &= \sum_{i=1}^{n} \frac{N_i}{z - a_i} + \sum_{k=1}^{l} \frac{r_k}{z - x_k}\,,
\end{aligned} \tag{A.2}
$$

while the resolvent $R(z)$ vanishes. In the first equation in (A.2), the occupation number r_k always vanishes unless $(f, \widetilde{f}) = (N_f, N_f)$.

By considering a generalized version of the Konishi anomaly and the factorization property of the chiral ring, we can write down the following algebraic equations for the generators of the chiral ring (2.3)

$$
\begin{aligned}
[V'(z)R(z)]_- &= R(z)^2\,, \\
[M(z)m(z)]_- &= R(z)\,, \\
[V'(z)T(z)]_- + \mathrm{tr}[m'(z)M(z)]_- &= 2R(z)T(z)\,,
\end{aligned} \tag{A.3}
$$

where the subscript means that we are dropping the non-negative powers in the Laurent expansion. By solving the first equation we learn that

$$
2R(z) = V'(z) - \sqrt{V'(z)^2 + f(z)}\,, \tag{A.4}
$$

where $f(z)$ is a degree $n-1$ polynomial with vanishing classical limit. This defines the curve of the gauge theory to be the hyperelliptic Riemann surface $y^2 = V'(z)^2 + f(z)$, which is a double-sheeted cover of the plane.

The solution of the second equation in (A.3) gives the meson generator $M(z)$, whose boundary conditions depend on the classical vacuum (A.1) we have chosen

$$
M(z) = \frac{R(z)}{m(z)} - \sum_{k=1}^{l} \frac{r_k V'(x_k) + (1 - 2r_k)R(x_k)}{z - x_k} \frac{1}{m'(x_k)}\,, \tag{A.5}
$$

where we suppressed the flavor indices, but we keep in mind that r_k can be nonzero only along the $(f, \widetilde{f}) = (N_f, N_f)$ flavor direction. The picture that emerges from (A.5) is that

JHEP11(2004)020

an Higgs eigenvalue in (A.1), i.e. $r_k = 1$, corresponds in the quantum theory to a pole for $M(z)$ in the first semiclassical sheet of the curve at x_k, while whenever $r_k = 0$ we have a pole in the second sheet.

The solution to the third equation in (A.3) again depends on the boundary conditions (A.1)

$$T(z) = \sum_{k=1}^{l} \frac{1}{2(z-x_k)} - \sum_{k=1}^{l} \frac{(1-2r_k)y(x_k)}{2y(z)(z-x_k)} + \frac{c(z)}{y(z)}\,, \tag{A.6}$$

where $c(z)$ is another degree $n-1$ polynomial.

B. Some properties of the effective glueball superpotential

In this appendix we will consider some properties of the coefficients h_i introduced in (2.69) and, by using these expressions, we will prove (2.70) up to an assumption of integrability.

B.1 Properties of the h_i

Consider the function $V(a_i)$ of the couplings t_j defined as

$$V(a_i) = \sum_{j=1}^{n} \frac{t_j}{j+1} a_i^{j+1}\,, \tag{B.1}$$

where a_i is solution of $V'(a_i) = 0$. Note that we are considering the $a_i = a_i(t_j)$ as functions of the couplings. Taking a derivative of $V(a_i)$ with respect to t_k we obtain

$$\frac{\partial V(a_i)}{\partial t_k} = \frac{a_i^{k+1}}{k+1}\,, \tag{B.2}$$

the second term in taking the derivative vanishing since it is multiplied by $V'(a_i)$. Since (B.2) is a derivative, it fulfills the condition

$$\frac{\partial}{\partial t_l} \frac{a_i^{k+1}}{k+1} = \frac{\partial}{\partial t_k} \frac{a_i^{l+1}}{l+1}\,, \tag{B.3}$$

and therefore

$$\frac{\partial}{\partial t_{n-l}} \frac{a_i^{j+l}}{j+l} = \frac{\partial}{\partial t_{n-k}} \frac{a_i^{j+k}}{j+k}\,. \tag{B.4}$$

which is our classical integrability condition.

Now we will assume that also the effective superpotential (2.70) satisfies the integrability condition

$$\frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial t_l \partial t_j} = \frac{\partial^2 \mathcal{W}_{\text{eff}}}{\partial t_j \partial t_l}\,. \tag{B.5}$$

By using the classical integrability (B.4), we find the relation

$$\sum_{i=1}^{n} \frac{\partial h_i}{\partial t_l} \frac{a_i^{j+1}}{j+1} = \sum_{i=1}^{n} \frac{\partial h_i}{\partial t_j} \frac{a_i^{l+1}}{l+1}\,. \tag{B.6}$$

Note that this relation will hold also for the d_i defined in (2.72)

$$\sum_{i=1}^{n} \frac{\partial d_i}{\partial t_l} \frac{a_i^{j+1}}{j+1} = \sum_{i=1}^{n} \frac{\partial d_i}{\partial t_j} \frac{a_i^{l+1}}{l+1} \,. \tag{B.7}$$

Finally, let us consider a scaling argument on the coefficients $h_i = h_i(t_k, N_l, S_j)$. Since

$$N_c = \sum_{i=1}^{n} h_i = \oint_A \frac{c(z)}{\sqrt{V'(z)^2 + f(z)}} \,, \tag{B.8}$$

if we rescale the glueballs $S_i \to \lambda S_i$ and the couplings $t_k \to \lambda t_k$, we have correspondingly that $V'(z) \to \lambda V'(z)$ and $f(z) \to \lambda^2 f(z)$, while the N_i are unchanged in (2.36). But by (B.8) also the h_i are invariant under the scaling, meaning that they are homogeneous functions of the couplings and the glueballs

$$\sum_{i=1}^{n} \left(t_i \frac{\partial}{\partial t_i} + S_i \frac{\partial}{\partial S_i} \right) h_l = 0 \,, \tag{B.9}$$

and this property carries on to the d_i.

B.2 Evaluation of $\mathcal{W}_{\text{eff}}$

We will prove, up to the assumption (B.5), that, in the notations of section 2.5, if we define

$$\mathcal{W}_{\text{eff}} = \sum_{i=1}^{n} d_i V(a_i) \,, \tag{B.10}$$

then we have

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial t_j} = \frac{1}{j+1} \sum_{i=1}^{n} h_i a_i^{j+1} \,. \tag{B.11}$$

Let us differentiate (B.10)

$$\frac{\partial \mathcal{W}_{\text{eff}}}{\partial t_j} = \sum_{i=1}^{n} \frac{d_i}{j+1} a_i^{j+1} + \sum_{i,k=1}^{n} t_k \frac{\partial d_i}{\partial t_j} \frac{a_i^{k+1}}{k+1} + \sum_{i=1}^{n} d_i \frac{\partial a_i}{\partial t_j} \sum_{k=1}^{n} t_k a_i^k \,, \tag{B.12}$$

but the last term vanishes since $V'(a_i) = 0$. Now we need to evaluate $\partial d_i / \partial t_j$. First note that $\partial_y d_i = d_i - h_i$ where $y = \sum_{i=1}^{n} \log S_i$. Then by using the homogeneity (B.9) of the d_i we have that

$$\sum_{k=1}^{n} t_k \frac{\partial d_i}{\partial t_k} = -\partial_y d_i \,. \tag{B.13}$$

Then we can use the integrability condition (B.7) for the second term in (B.12) and get

$$\begin{aligned} \frac{\partial \mathcal{W}_{\text{eff}}}{\partial t_j} &= \sum_{i=1}^{n} d_i \frac{a_i^{j+1}}{j+1} - \sum_{i=1}^{n} \frac{\partial d_i}{\partial y} \frac{a_i^{j+1}}{j+1} \\ &= \sum_{i=1}^{n} d_i \frac{a_i^{j+1}}{j+1} - \sum_{i=1}^{n} (d_i - h_i) \frac{a_i^{j+1}}{j+1} \\ &= \sum_{i=1}^{n} h_i \frac{a_i^{j+1}}{j+1} \,. \end{aligned} \tag{B.14}$$

C. Several Higgs solution

In this appendix we will generalize the Higgs solution (3.7) to the case in which more than one electric color direction is higgsed on the same electric flavor.

C.1 Two-Higgs case

The electric theory. Consider the electric theory with superpotential

$$W_{el} = \frac{t_n}{n+1}\,\mathrm{Tr}\,X^{n+1} + m_3\widetilde{Q}_{N_f}X^2Q^{N_f} + m_2\widetilde{Q}_{N_f}XQ^{N_f} + m_1Q_{N_f}Q^{N_f}\,. \tag{C.1}$$

We can get a classical vacuum in which the gauge group is higgsed as $\mathrm{U}(N_c) \to \mathrm{U}(N_c-2)$ by considering the following expectation values

$$\begin{aligned} X &= \mathrm{diag}(x_1, x_2, 0, \ldots, 0) \\ \widetilde{Q}_{N_f} &= (\widetilde{h}_1, \widetilde{h}_2, 0, \ldots, 0), \quad Q^{N_f} = (h_1, h_2, 0, \ldots, 0)\,, \end{aligned} \tag{C.2}$$

where $\widetilde{h}_i h_i = -V'(x_i)/m'(x_i)$. We denoted by $x_{1,2}$ the two roots of the meson polynomial $m_3z^2 + m_2z + m_1$. As is well known, the roots of a quadratic algebraic equation satisfy

$$-\,(x_1+x_2) = \frac{m_2}{m_3}\,, \qquad x_1x_2 = \frac{m_1}{m_3}\,. \tag{C.3}$$

The magnetic theory. The superpotential for the magnetic theory is

$$W_{\mathrm{mag}} = -\frac{t_n}{n+1}\,\mathrm{Tr}\,X^{n+1} + \widetilde{q}\widetilde{m}(P,Y)q + m_1(P_1)^{N_f}_{N_f} + m_2(P_2)^{N_f}_{N_f} + m_3(P_3)^{N_f}_{N_f}\,. \tag{C.4}$$

By (3.11), the singlets acquire the classical expectation value $P_j = \widetilde{h}_1h_1x_1^{j-1} + \widetilde{h}_2h_2x_2^{j-1}$ corresponding to the electric mesons.

We expected the magnetic gauge group to break down to $\mathrm{U}(\bar{N}_c) \to \mathrm{U}(\bar{N}_c - n + 2)$, so that the vev for the adjoint will be a nonvanishing block of rank $n-2$. By using the property (C.3) of the roots of $m(x)$ we can find the solution to the singlet, fundamental and adjoint equations of motion. In the notations of (2.15), the only nonvanishing entries in the adjoint are

$$\begin{aligned} Y &= |1\rangle\langle v_2| + b_3R_{n-2}\,, \\ |v_2\rangle &= -\sum_{j=1}^{n-2}\frac{1}{b_3}^{j-1}\Bigg[\left(-\frac{m_2}{m_3}\right)^j + \sum_{k=1}^{[j/2]}(j-k)\left(-\frac{m_2}{m_3}\right)^{j-2k}\left(-\frac{m_1}{m_3}\right)^k + \\ &\qquad + \delta^j_{\mathrm{even}}\binom{j}{2}\left(-\frac{m_1}{m_3}\right)^{[j/2]}\Bigg]|j\rangle\,, \end{aligned} \tag{C.5}$$

while the fundamentals are $|q_{N_f}\rangle = b_3|n-2\rangle$ and $|\widetilde{q}^{N_f}\rangle = b_3|1\rangle$ and b_3 is defined in (2.12).

C.2 Several Higgs

The electric theory. Consider the electric theory with a generic meson polynomial on the last flavor

$$\begin{aligned} W_{el} &= \frac{t_n}{n+1}\operatorname{Tr} X^{n+1} + \widetilde{Q}_{N_f} m(X) Q^{N_f}\,, \\ m(x) &= \sum_{k=1}^{l+1} m_k x^{k-1}\,. \end{aligned} \tag{C.6}$$

The polynomial $m(x)$ has l roots that we denote $x_1,\ldots,x_l$. The following property between the coefficients of the polynomial and its roots holds

$$\frac{m_{l+1-i}}{m_{l+1}} = (-)^i \sum_{k_1<k_2<\cdots<k_i} x_{k_1}x_{k_2}\cdots x_{k_i}\,. \tag{C.7}$$

We can consider the following vacuum

$$\begin{aligned} X &= \operatorname{diag}(x_1,x_2,\ldots,x_l,0,\ldots,0) \\ \widetilde{Q}_{N_f} &= (\widetilde{h}_1,\widetilde{h}_2,\ldots,\widetilde{h}_l,0,\ldots,0)\,, \qquad Q^{N_f} = (h_1,h_2,\ldots,h_l,0,\ldots,0)\,, \end{aligned} \tag{C.8}$$

where $\widetilde{h}_i h_i = -V'(x_i)/m'(x_i)$. Note that each root x_k can appear just once in the adjoint expectation value. In this way we break the gauge symmetry as $\mathrm{U}(N_c) \to \mathrm{U}(N_c - l)$. We can higgs at most $n-1$ colors on the same flavor, corresponding to the largest degree the meson polynomial $m(z)$ can have.

The magnetic theory. According to the above discussion, in the magnetic theory we will have to solve the singlet equations of motion

$$\begin{aligned} \widetilde{q}^{N_f} Y^{n-i} q_{N_f} &= \frac{m_i}{m_{l+1}} b_{l+1}^{n-l+1}\,, \qquad i = 1,\ldots,l+1\,, \\ \widetilde{q}^{N_f} Y^j q_{N_f} &= 0\,, \qquad j = 0,\ldots,n-l-2\,, \end{aligned} \tag{C.9}$$

where we used (C.7) to ease the notation and b_{l+1} is defined in (2.12). The solution for the adjoint, which generalizes (C.5), can be sketched as the nonvanishing block of rank $n-l$

$$\begin{aligned} Y &= |1\rangle\langle v_l| + b_{l+1} R_{n-l}\,, \\ |v_l\rangle &= -\sum_{j=1}^{n-l} \frac{1}{b_{l+1}}^{j-1} \left[\left(-\frac{m_l}{m_{l+1}}\right)^j + \cdots\right] |j\rangle\,, \\ |q_{N_f}\rangle &= b_{l+1}|n-l\rangle, \qquad |\widetilde{q}^{N_f}\rangle = b_{l+1}|1\rangle\,, \end{aligned} \tag{C.10}$$

where the dots stand for an expression analogous to the one in (C.5) but more involved. In this way we break the magnetic gauge group down to $\mathrm{U}(\bar{N}_c - n + l) = \mathrm{U}(n(N_f-1)-(N_c-l))$. Note that this solution holds only for $l \le n-1$, as we saw on the electric side.

References

[1] N. Seiberg, *Electric-magnetic duality in supersymmetric nonabelian gauge theories*, *Nucl. Phys.* **B 435** (1995) 129 [hep-th/9411149].

[2] K.A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality*, *Nucl. Phys.* **45BC** *(Proc. Suppl.)* (1996) 1 [hep-th/9509066].

[3] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255]; *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106]; *A perturbative window into non-perturbative physics*, hep-th/0208048.

[4] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[5] K. Konishi, *Anomalous supersymmetry transformation of some composite operators in sqcd*, *Phys. Lett.* **B 135** (1984) 439;
K. Konishi and K. Shizuya, *Functional integral approach to chiral anomalies in supersymmetric gauge theories*, *Nuovo Cim.* **A90** (1985) 111.

[6] B. Feng, *Seiberg duality in matrix model*, hep-th/0211202;
B. Feng and Y.-H. He, *Seiberg duality in matrix models, II*, *Phys. Lett.* **B 562** (2003) 339 [hep-th/0211234];
B. Feng, *Note on Seiberg duality in matrix model*, *Phys. Lett.* **B 572** (2003) 68 [hep-th/0303144].

[7] D. Kutasov, *A comment on duality in $N = 1$ supersymmetric nonabelian gauge theories*, *Phys. Lett.* **B 351** (1995) 230 [hep-th/9503086].

[8] D. Kutasov and A. Schwimmer, *On duality in supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 354** (1995) 315 [hep-th/9505004].

[9] D. Kutasov, A. Schwimmer and N. Seiberg, *Chiral rings, singularity theory and electric-magnetic duality*, *Nucl. Phys.* **B 459** (1996) 455 [hep-th/9510222].

[10] N. Seiberg, *Adding fundamental matter to 'chiral rings and anomalies in supersymmetric gauge theory'*, *J. High Energy Phys.* **01** (2003) 061 [hep-th/0212225].

[11] F. Cachazo, N. Seiberg and E. Witten, *Chiral rings and phases of supersymmetric gauge theories*, *J. High Energy Phys.* **04** (2003) 018 [hep-th/0303207].

[12] I. Affleck, M. Dine and N. Seiberg, *Dynamical supersymmetry breaking in supersymmetric QCD*, *Nucl. Phys.* **B 241** (1984) 493; *Dynamical supersymmetry breaking in four-dimensions and its phenomenological implications*, *Nucl. Phys.* **B 256** (1985) 557.

[13] C. Csáki and H. Murayama, *Instantons in partially broken gauge groups*, *Nucl. Phys.* **B 532** (1998) 498 [hep-th/9804061]; *New confining $N = 1$ supersymmetric gauge theories*, *Phys. Rev.* **D 59** (1999) 065001 [hep-th/9810014].

[14] M. Klein and S.-J. Sin, *Matrix model, kutasov duality and factorization of Seiberg-Witten curves*, hep-th/0310078.

[15] R. Argurio, G. Ferretti and R. Heise, *An introduction to supersymmetric gauge theories and matrix models*, *Int. J. Mod. Phys.* **A 19** (2004) 2015 [hep-th/0311066].

[16] F. Cachazo and C. Vafa, *$N = 1$ and $N = 2$ geometry from fluxes*, hep-th/0206017.

JHEP11(2004)020

[17] L.F. Alday and M. Cirafici, *Effective superpotentials via konishi anomaly*, *J. High Energy Phys.* **05** (2003) 041 [hep-th/0304119].

[18] C.-H. Ahn, B. Feng, Y. Ookouchi and M. Shigemori, *Supersymmetric gauge theories with flavors and matrix models*, *Nucl. Phys.* **B 698** (2004) 3 [hep-th/0405101].

[19] F. Cachazo, *Notes on supersymmetric* $\mathrm{Sp}(N)$ *theories with an antisymmetric tensor*, hep-th/0307063;
M. Matone, *The affine connection of supersymmetric* $\mathrm{SO}(N)/\mathrm{Sp}(N)$ *theories*, *J. High Energy Phys.* **10** (2003) 068 [hep-th/0307285].

[20] J.H. Brodie, *Duality in supersymmetric* $\mathrm{SU}(N_c)$ *gauge theory with two adjoint chiral superfields*, *Nucl. Phys.* **B 478** (1996) 123 [hep-th/9605232].

[21] K. Intriligator and B. Wecht, *The exact superconformal R-symmetry maximizes a*, *Nucl. Phys.* **B 667** (2003) 183 [hep-th/0304128].

[22] D. Kutasov, A. Parnachev and D.A. Sahakyan, *Central charges and* $\mathrm{U}(1)_R$ *symmetries in* $N=1$ *super Yang-Mills*, *J. High Energy Phys.* **11** (2003) 013 [hep-th/0308071].